PACIFIC JOURNAL OF MATHEMATICS
Vol. 83, No. 2, 1979

A RADON-NIKODYM THEOREM FOR FINITELY ADDITIVE BOUNDED MEASURES

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An exact Radon-Nikodym theorem is obtained for finitely additive bounded scalar measures defined on a field, the additional condition being a local condition on the dominant average range. The traditional technique of transferring the problem to the Stone space, which results in approximate Radon-Nikodym derivatives, is circumvented by isolating an Exhaustion principal for finitely additive measures which is then utilized to obtain the necessary decompositions.

Examples are given to illustrate the basic difficulties which arise in differentiating with respect to signed finitely additive measures and it is demonstrated that one difficulty arises from a lack of a suitable Hahn decomposition of the differentiating measures. The concept of an exhaustive Hahn decomposition is defined for finitely additive measures and is compared to the related concepts of an approximate Hahn decomposition as well as the standard Hahn decomposition. It is shown that μ having an exhaustive Hahn decomposition is equivalent to $|\mu|$ having a Radon-Nikodym derivative with respect to μ and this result is then applied, in this situation, to obtain a simplified Radon-Nikodym theorem.

The question of characterizing indefinite integrals of finitely additive measures has been under consideration for a number of years. There have been two basic approaches to this problem, both seemingly arising from a desire to characterize the absolutely continuous bounded measures. The first was to enlarge the class of integrable functions to include objects other than point functions and to then obtain an equivalence between absolute continuity and integral representation. Rickart [10] obtained such an equivalence by including the multi-valued contractive set functions, while Tucker and Wayment [12], in the setting of finitely additive operator-valued measures, obtained a similar equivalence between an enlarged class of integrable objects and a generalized definition of absolute continuity. The second approach is that of the Radon-Nikodym Bochner theorem [3, p. 315, Theorem 14] which utilized the Stone space to characterize the absolutely continuous, bounded variation measures as those which can be approximated arbitrarily close in variation by integrals of integrable simple functions. There does not seem to be any characterization of indefinite integrals of point functions with respect to a finitely additive bounded scalar measure prior to