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LONG WALKS IN THE PLANE WITH FEW COLLINEAR POINTS

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Let S be a set of vectors in R^n. An S-walk is any (finite or infinite) sequence $\{z_i\}$ of vectors in R^n such that $z_{i+1}-z_i \in S$ for all i. We will show that if the elements of S do not all lie on the same line through the origin, then for each integer $K \geqq 2$, there exists an S-walk $W_K=\{z_i\}_{i=1}^{N(K)}$ such that no $K+1$ elements of W_K are collinear and $N(K)$ grows faster than any polynomial function of K.

Specifically, we will prove that

$$\log_2 N(K) > \frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) .$$

We will then show that if the elements of S lie on at least L distinct lines through the origin, then there exists an S-walk of length $N(K, L)$ with no $K+1$ elements collinear, such that $N(K, L) \geqq (1/4)L^* N(K-1)$, where $L-2 \leqq L* \leqq L+1$ and $L* \equiv 0 \bmod 4$. In [3] it was shown that if $S \subset Z^2$, and for all $s \in S$ we have $\|s\| \leqq M$, then there does not exist an S-walk $W = \{z_i\}_{i=1}^{N(K,M)}$ such that no $K+1$ elements of W are collinear and

$$\log_2 N(K, M) > 2^{13} M^4 K^4 + \log_2 K .$$

Before proving these theorems we introduce some notation. If $A = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$ and $B = (\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m)$ are ordered sets of vectors, we let $RA = (\boldsymbol{a}_n, \cdots, \boldsymbol{a}_1)$ and we let $(A, B) = (\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_m)$. We let $2A = (A, A)$ and, for every positive integer k, we let $(k+1)A = (kA, A)$. If J is a vector operator, we let $JA = (J\boldsymbol{a}_1, \cdots, J\boldsymbol{a}_n)$.

THEOREM 1. *Let S contain two vectors independent over $\boldsymbol{R}$, and let K be an integer greater than or exual to 2. There exists an S-walk $W_K = \{z_p\}_{p=1}^{N(K)}$ such that no $K+1$ elements of W_K are collinear and such that*

$$\log_2 N(K) > \frac{1}{9}(\log_2 K - 1)^2 - \frac{1}{6}(\log_2 K - 1) .$$

Proof. If we let $(\log_2 K - 1)^2/9 - (\log_2 K - 1)/6 = \log_2 K$, then $\log_2 K = (25 + 3\sqrt{65})/4 > 12$ or $(25 - 3\sqrt{65})/4 < 1$. Therefore if $1 \leqq \log_2 K \leqq 12$, and $2 \leqq K \leqq 4096$, then