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LOCALE GEOMETRY

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We commence with a locale $\mathscr{L}$ (that is, a complete Heyting algebra) and introduce the notion of an $\mathscr{L}$-valued betweenness relation on a set. The concept of an $\mathscr{L}$-valued geometry is then formulated and the relevant versions of the Radon, Helly and Carathéodory theorems are proved.

Introduction. The abstract theory of join systems was developed by W. Prenowitz [8] and [9] as an aid to studying descriptive and spherical geometries. This notion of join system has since been further developed by V. W. Bryant and R. J. Webster [1] to enable the corresponding axiomatic treatment of such results as the Radon, Helly and Carathéodory theorems. It is this aspect of the theory with which the present article is concerned.

We commence this article by extending the notion of a join system so that it is no longer necessarily two-valued. More precisely, given a locale lattice $\mathscr{L}$, we introduce the notion of an $\mathscr{L}$-valued betweenness relation $(-, -, -): X \times X \times X \to \mathscr{L}$ on a set X; if $(x, y, z) = p \in \mathscr{L}$ we might say that the point z lies on the segment (x, y) with "probability p". This loose description is related to theories of multivalued logic which arise in topos theory. Indeed, one can develop join systems in a reasonably complete topos in terms of multivalued join systems over the category of sets; see §4. These notions, in turn, give rise to the forms of the Radon, Helly and Carathéodory theorems dicussed in §3.

We emphasize here that, in this preliminary article, we do not deal with multigroups (after W. Prenowitz) nor do we enter into all aspects of dimension theory (after V. W. Bryant and R. J. Webster). Also we leave the proof of the more basic elementary deductions as simple exercises for the reader; these results are used without reference.

1. $\mathscr{L}$-forms.

Let $\mathscr{L}$ be a locale and let X be a set. A *symmetric $\mathscr{L}$-form* on X is a function $X(-, -): X \times X \to \mathscr{L}$ such that $X(x, x) = 1$, $X(x, y) = X(y, x)$, $\sup_y X(x, y) \wedge X(y, z) = X(x, z)$. A *functional* on X is a set map $A: X \to \mathscr{L}$ such that $A = \sup_x A(x) \wedge X(x, -)$. A *singleton*, or *point* is a functional of the form $\bar{x} = X(x, -): X \to \mathscr{L}$. Thus each functional is an "expansion of singletons" or an "internal colimit of points". For notational convenience we shall represent $\bar{x}$ simply by x unless we wish to emphasize the distinction.

The ordered set of functionals on X is denoted $\mathrm{Fnl}(X, \mathscr{L})$; it