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AN ANTI-OPEN MAPPING THEOREM FOR FRÉCHET SPACES

STEVEN F. BELLENOT

It is well-known that completeness is necessary for the usual open mapping theorem for Fréchet spaces. In contrast, it is shown that, with the obvious exception of ω, each infinite-dimensional Fréchet space has another distinct complete topology with the same continuous dual.

By a space or subspace, we mean an infinite-dimensional locally convex Hausdorff topological vector space over either the real or the complex scalars. Our notation generally follows Robertson and Robertson [7]. In particular, X' and $\sigma(X, X')$ denote the continuous dual and the weak topology on X, respectively. Denote by ω (respectively, ϕ) the space formed by the product (respectively, direct sum) of countably-many copies of the scalar field. We use c_0, l_1 and l_∞ to denote the Banach sequence spaces (with their usual norms) of, respectively, null sequences, absolutely summable sequences and bounded sequences.

Our main result can be stated as:

THEOREM. *Each Fréchet space $(X, \zeta) \neq \omega$ has a topology η, so that, $\sigma(X, X') < \eta < \zeta$ and the space (X, η) is complete.*

By the open mapping theorem, (X, η) is a complete space which is not barrelled. In Section one we prove the theorem for the special cases of $(X, \zeta) = c_0$ (Case I) and (X, ζ) a nuclear space with a continuous norm (Case II). Then in Section two we reduce the theorem to these special cases.

We will have occasion to use Grothendieck's characterization of the completion of the space (X, ζ) as the set of linear functionals on X' which are $\sigma(X', X)$-continuous on ζ-equicontinuous sets (see Robertson and Robertson [7], p. 103). Berezanskii's [4] (see also [2, pp. 61–62]) notion of inductive semi-reflexivity is used in Case II. In particular, complete nuclear spaces are inductive semi-reflexive, and the topology constructed from $\{\mu_n\}$ in Case II is complete in any inductive semi-reflexive space. The only other fact used about nuclear spaces is that their topology can be defined by means of (semi-) inner products (see Case II and Schaefer [7] p. 103).

Perhaps it is worth pointing out, that there are always lots of differently-defined complete topologies on each complete separable space (see Bellenot [1], [2] and with Ostling [3]): the difficulty is in