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ON A THEOREM OF HAYMAN CONCERNING THE DERIVATIVE OF A FUNCTION OF BOUNDED CHARACTERISTIC

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W. Hayman [On Nevanlinna's second theorem and extensions, Rend. Circ. Mat. Palermo, Ser. II, II (1953).] has given sufficient conditions on a function, f, of bounded characteristic in the unit disc, in order that f' also have bounded characteristic. In this paper it is shown that one of these conditions is also necessary for the conclusion of the theorem to hold.

Let U be the open unit disc in the complex plane and let T be its boundary. It is well known that there are functions f, that are bounded and holomorphic in U, such that $f' \notin N(U)$. Here $N(U)$ is the Nevanlinna class. In fact, O. Frostman, [1, Théoreme IX], has shown that there are Blaschke products with some degree of "smoothness" whose derivatives fail to lie in $N(U)$. More precisely, he shows that there is a Blaschke product B, whose zeros $\{a_n\}$ satisfy the condition,

$$\sum_n (1 - |a_n|)^\alpha < \infty , \quad \text{for all} \quad \alpha > \frac{1}{2} ,$$

but $B' \notin N(U)$. In Frostman's example, every point of T is a limit point of the sequence $\{a_n\}$.

W. Hayman, [2, Theorem IV], has proved a result in the positive direction. A function f, that is holomorphic in a bounded domain D, is said to be of order K if, for every complex number a, the number of solutions of the equation, $f(z) = a$, that are at a distance of at least ε from the boundary of D is at most $C\varepsilon^{-K}$, for some constant C. C may depend on a but not on ε. We say f has finite order if it has order K for some K. Now let D be a bounded open set such that $U \subsetneqq D$, and let $D \cap T = \bigcup_n I_n$, where $I_n = \{e^{i\theta}: \alpha_n < \theta < \beta_n\}$.

THEOREM A (*Hayman*). *Suppose that*

(i) (a) $\sum_n (\beta_n - \alpha_n) = 2\pi$

(b) $\sum_n (\beta_n - \alpha_n) \log 1/(\beta_n - \alpha_n) < \infty$.

(ii) *there are constants* $\varepsilon, C > 0$ *such that if* $\alpha_n < \theta < \beta_n$, *then*

$$\operatorname{dist}(e^{i\theta}, \partial D) \geqq \varepsilon(|\theta - \alpha_n|\,|\theta - \beta_n|)^C .$$

(iii) f *is holomorphic and of finite order in* D *and* $f \in N(U)$.