FACIFIC JOURNAL OF MATHEMATICS
Vol. 84, No. 1, 1979

GENERALIZATION OF A THEOREM OF LANDAU

MIRIAM HAUSMAN

A well known theorem of Landau asserts that

$$\underline{\lim}_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-\gamma} \tag{1.1}$$

where γ = Euler's constant. In this paper a generalization is obtained by focusing on

$$G(k) = \underline{\lim}_{n\to\infty} (\log\log n)^{1/k} \max\left(\frac{\phi(n+1)}{n+1}, \cdots, \frac{\phi(n+k)}{n+1}\right). \tag{1.2}$$

Clearly, the assertion $G(1) = e^{-\gamma}$ is precisely Landau's theorem. It is proved that

$$G(k) = e^{-\gamma/k} \prod_{p<k}\left(1 - \frac{1}{p}\right)^{-1/k} \psi(k) \tag{1.3}$$

where

$$\psi(k) = \prod_{\substack{p \mid k \\ p<k}}\left(1 - \frac{1}{p}\right)^{1/p} \prod_{\substack{p \nmid k \\ p<k}}\left(1 - \frac{1}{p}\right)^{(1/k)[k/p]+1/k}. \tag{1.4}$$

The function $\psi(k)$ satisfies $0 < \psi(k) \leqq 1$ and it is easily seen from (1.4) that

$$\lim_{k\to\infty} \psi(k) = \prod_{p}\left(1 - \frac{1}{p}\right)^{1/p}. \tag{1.5}$$

2. Preliminary lemmas. The results obtained in this paper depend on the following well known theorems [1], [2], and [3].

$$\underline{\lim}_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-\gamma} \qquad \text{(Landau's theorem)} \tag{2.1}$$

$$\sum_{p\leqq x} \frac{1}{p} = \log\log x + c + O\left(\frac{1}{\log x}\right) \qquad \text{(Mertens')} \tag{2.2}$$

$$\prod_{p\leqq x}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O\left(\frac{1}{\log^2 x}\right) \qquad \text{(Mertens')} \tag{2.3}$$

3. Proof of (1.3). We introduce

$$\left(\frac{\phi(n)}{n}\right)_k = \prod_{\substack{p \mid n \\ p\geqq k}}\left(1 - \frac{1}{p}\right) \tag{3.1}$$

and

$$f_k(n) = \prod_{\substack{p \mid n \\ p<k}}\left(1 - \frac{1}{p}\right) \tag{3.2}$$