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VALENCE PROPERTIES OF THE SOLUTION OF A DIFFERENTIAL EQUATION

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Libera proved that the first order linear differential equation $F(z)+zF'(z)=2f(z)$ has a convex, starlike or close-to-convex solution in $|z|<1$ if the driving term $f(z)$ is convex, starlike, or close-to convex in $|z|<1$. It was an open question whether the solution would be univalent if $f(z)$ were spiral-like or univalent. The paper shows the relation of Libera's question to the Mandelbrojt — Schiffer conjecture and the class M defined by S. Ruscheweyh. The paper proves there are spiral-like functions $f(z)$ for which the solution of the above differential equation is of infinite valence. The paper closes with four open problems.

Libera [6] proved that if $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ maps $|z|<1$ onto a convex, starlike, or close-to-convex domain, then so does $F(z)=2z^{-1}\int_0^z f(t)dt=z+\sum_{n=2}^{\infty}2a_nz^n/(n+1)$. Bernardi [1] then proved that if $f(z)$ maps $|z|<1$ onto a convex, starlike, or close-to-convex domain, then for any positive integer c, $F_c(z)=(c+1)z^{-c}\int_0^z t^{c-1}f(t)dt=\sum_{n=1}^{\infty}(c+1)a_nz^n/(n+c)$ does also. Lewandowski, Miller, and Zlotkiewicz noted that Bernardi's result could be rephrased as, for any positive integer c, the first order linear differential equation

$$cF(z)+zF'(z)=(c+1)f(z) \tag{1}$$

with convex, starlike, or close-to-convex driving term $f(z)$ has a convex, starlike, or close-to-convex solution. They then proved [5] that (1) has a starlike univalent solution for any starlike driving function $f(z)$ for any complex c with $\operatorname{Re} c\geqq 0$.

Libera [8, Problem 2.3] asked whether the differential equation (1) would have this geometric invariance property if $f(z)$ were univalent or if $f(z)$ were spiral-like. Before we answer both of these questions in the negative, let us see how his question is connected with the Mandelbrojt-Schiffer conjecture for univalent functions.

Mandelbrojt and Schiffer conjectured that if $f(z)=\sum a_nz^n$ and $g(z)=\sum b_nz^n$ are univalent in $|z|<1$, then so also are the functions $H^*=\{f*g(z): f*g(z)=\sum a_nb_nz^n/n\}$. This was settled negatively (it would have implied the Bieberbach conjecture) in three separate papers. Hayman [4] exhibited a univalent function $f(z)$ such that $f*f(z)$ grows too fast for z near 1. His analysis shed no light on the valence of functions in H^*. Epstein and Schoenberg [2] exhibited