PACIFIC JOURNAL OF MATHEMATICS
Vol. 84, No. 2, 1979

# RIESZ-PRESENTATION OF ADDITIVE AND $\sigma$-ADDITIVE SET-VALUED MEASURES

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**In this paper we generalize the well known Riesz's representation theorems for additive and $\sigma$-additive scalar measures to the case of additive and $\sigma$-additive set-valued measures.**

**1. Introduction.** Consider a nonvoid set $\Omega$ and an algebra $\mathscr{A}$ over $\Omega$. *An additive set-valued measure* $\Phi$ on the field $(\Omega, \mathscr{A})$ is a function $\Phi: \mathscr{A} \to \{T \subset \boldsymbol{R}^m: T \neq \emptyset\}$ from $\mathscr{A}$ into the class of all non-empty subsets of $\boldsymbol{R}^m$, which is additive, i.e.,

$$\emptyset \neq \Phi(A) \subset \boldsymbol{R}^m \quad \text{for all} \quad A \in \mathscr{A}$$

and

$$\Phi(A_1 \cup A_2) = \Phi(A_1) + \Phi(A_2)$$

for every pair of disjoint sets $A_1, A_2 \in \mathscr{A}$. If $\mathscr{A}$ is a $\sigma$-algebra then $\Phi$ is called a *$\sigma$-additive set-valued measure*, iff

$$\Phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Phi(A_n)$$

for every sequence $A_1, A_2, \cdots$ of mutually disjoint elements of $\mathscr{A}$. Here the sum $\sum_{n=1}^{\infty} T_n$ of the subsets $T_1, T_2, \cdots$ of $\boldsymbol{R}^m$ consists of all thevectors: "$x = \sum_{n=1}^{\infty} x_n$ with $x_n \in T_n$ for $n \in \boldsymbol{N}$. In the sequel, "$\Phi | \mathscr{A}$ is an additive [resp. $\sigma$-additive] set-valued measure" is an abbreviation for an algebra [resp. a $\sigma$-algebra] over $\Omega$ and a function $\Phi: \mathscr{A} \to \{T \subset \boldsymbol{R}^m: T \neq \emptyset\}$ which is additive [resp. $\sigma$-additive]. The calculus of additive and $\sigma$-additive set-valued measures has recently been developed by several authors (see [2], [4], [5], [1] and [6]) and the ideas and techniques have many interesting applications in mathematical economics (see [3], [4] and [10]), in control theory (see [8] and [9]), and other mathematical fields. Additive and $\sigma$-additive set-valued measures have also been discussed for their own mathematical interest, because they extend the theory of scalar additive and $\sigma$-additive measures in a natural way. This is the background of the present paper. Theorems 1 and 2 extend the known representation theorems of Riesz for bounded, additive [resp. regular, $\sigma$-additive] scalar measures to the case of bounded, additive [resp. regular, $\sigma$-additive] set-valued measures.