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INEQUALITIES INVOLVING DERIVATIVES

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This paper deals with generalizations of classical results on real-valued functions of a real variable which are of the following type: *Bounds for the function and for its* mth *derivative imply bounds for the* kth *derivative* $0 < k < m$. **Our theorems extend these results in various directions, the most important being the extension to functions of** n **variables.**

(A) The Hadamard-Littlewood three-derivatives theorem states that if $u(t) = o(1)$ and $u''(t) = O(1)$ as $t \to \infty$, then $u'(t) = o(1)$. In Theorem 1, the more general version "$u(t) = o(1)$ and $u^{(m+1)}(t) = O(1)$ implies $u^{(k)}(t) = o(1)$ for $1 \leqq k \leqq m$" is generalized in three directions. The assumption that $u = o(1)$ is weakened, the functions considered are Banach-space valued, and the boundedness of $u^{(m+1)}$ is replaced by a condition on $u^{(m)}$ which is weaker than uniform continuity. A similar result for functions of several variables is given in Theorem 4.

(B) Let $u(t)$ be of class C^m in an unbounded interval J and let

$$U_k = \sup_{t \in J} |u^{(k)}(t)| .$$

Inequalities of the form

$$U_k \leqq A(m, k) U_0^{1-k/m} U_m^{k/m} , \qquad 0 \leqq k \leqq m ,$$

hold for such functions, as is well known. In Theorem 5 we extend these inequalities to Banach-space valued functions $u(x)$ defined in suitably restricted domains of R^n. Counterexamples show that the restrictions imposed on the domain are appropriate.

(C) If J is an interval of finite length $|J|$, the inequality (B) is no longer valid. (It can be saved by imposing homogeneous boundary conditions, but this will not be done here.) We shall show that an inequality

$$U_k \leqq A(m, k) U_0^{1-k/m} (U_m^*)^{k/m} , \qquad 0 \leqq k \leqq m ,$$

still holds, where

$$U_m^* = \max(U_0 |J|^{-m}, U_m) .$$

In Theorem 2 this result is presented for Banach-space valued functions in bounded or unbounded domains of R^n.

It is not our aim to obtain the best or even good constants. In the one-dimensional case, the problem of finding the optimal constants