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INVARIANT MEANS AND ANALYTIC ACTIONS

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Let $T \times D \rightarrow D$ be a separately continuous analytic action of a semitopological semigroup T on D, the open unit disk in the complex plane, and let K be a compact T-invariant subset of D. The chief result of this paper is that if $AP(T)$, the space of almost periodic functions on T, has a left invariant mean, then D contains a common fixed-point of T. As a special case, we show that a finite group of analytic self maps of D has a common fixed-point in D.

M. M. Day [1] pioneered the investigation of the relationship between fixed-point properties of affine actions of a semigroup T to the existence of an invariant mean on T.

A. T. Lau [9] obtained the result that if $AP(T)$ has a left invariant mean, then every equicontinuous affine separately continuous action $T \times K \rightarrow K$ of a semitopological semigroup T on a compact convex subset K of a locally convex (separated) linear topological space has a common fixed-point in K. (A converse to this result was also obtained in [9].) Variants of this result concerning the fixed-point properties of actions of semitopological semigroups T, when $AP(T)$ has a left invariant mean, were studied later by J. C. S. Wong [14] and by H. D. Junghenn [7]. The results alluded to in this paragraph are united by a common theme in their proofs; they were shown essentially by "pulling back" various spaces of real valued functions on K into the space $AP(T)$.

However, there are certain fixed-point theorems about actions of T, when $AP(T)$ has a left invariant mean, which are not obtained by pull-back arguments alone, but which rely on some special structure theorem of the space on which T acts. As an example, in [10] Lau showed that if $T \times I \rightarrow I$ is an equicontinuous action of T on I, the closed unit interval of the real line, then I contains a common fixed-point of T. But the proof of this makes use of a fixed-point theorem of T. Mitchell [11, Theorem 2, p. 149], which in turn rests on the fact that a compact group of continuous self-maps of I has a common fixed-point in I (see [5, 3.24, p. 333]). However, this last fact no longer holds if I is replaced by an arbitrary disk K in n-space, even if the group is required to be finite and Abelian. (For counter-examples, see R. Oliver [12, Theorem 7, p. 174].) Moreover, the result of Lau in [10] no longer holds if I is replaced by such an arbitrary K.

Let f, g be two continuous commuting self-maps of $\bar{D}$, the closed unit disk in the complex plane, and let f, g be analytic on D.