PACIFIC JOURNAL OF MATHEMATICS
Vol. 85, No. 1, 1979

N-DIMENSIONAL AREA AND CONTENT IN MINKOWSKI SPACES

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A definition of content in Minkowski spaces (of any finite dimension) is given which implies that the surface of the unit ball and that of the dual ball are equal. Various consequences of this definition, including the solution to the isoperimetric problem, are explored. Numerous examples and some unsolved problems are given in the last two sections.

1. Introduction. The original motivation for this investigation came from the work of J. J. Schäffer on geometrical constants associated with the unit ball in a normed linear space. He and K. Sundaresan [15] showed that, if a normed linear space is nonreflexive, then the "girth" of the unit ball is 4. In the paper [13], in which girth and inner diameter were first defined, he also showed that, if $\mathscr{X}$ is 2-dimensional, the girth lies in the interval [6, 8]. It was these latter inequalities which suggested the consideration of higher dimensional parameters (area, volume, etc.) and, specifically, to ask for bounds for the surface area of the unit ball in a 3-dimensional space. This problem is still, as far as we know, unsolved (see Problem 7.9 below); but it requires, first of all, a definition of area. Such a definition is the first aim of the present paper, and our proposal is contained in § 2. The second aim is to investigate the solution to the isoperimetric problem which results from our definition of area; this is contained in § 4. Throughout this paper, but especially in § 4, we rely on the work of H. Busemann and his school [3], [4], [5], [6]. A more detailed summary of the contents of each section will be given after we have explained our notation.

Throughout, we will be concerned with finite-dimensional real linear spaces upon which we impose a variety of norms. Script roman letters will be used for affine sets — $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ for spaces and subspaces, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$ for hyperplanes. Small roman letters will be used for vectors — $p, q, \cdots, z$ in the space $\mathscr{X}$ and $a, b, \cdots, h$ in the dual space (i.e., linear functionals on $\mathscr{X}$). For obvious reasons it is not always possible to be completely systematic about this. Letters from the middle of the alphabet — $i, j, \cdots, n$ — will be reserved for natural numbers. Capital roman letters — $A, B, \cdots$ — will be used to denote convex sets, while nonconvex sets (and surfaces in particular), will be denoted by capital greek letters