PACIFIC JOURNAL OF MATHEMATICS
Vol. 85, No. 2, 1979

HOMOTOPY THEORY OF RIGID PROFINITE SPACES I.

D. L. RECTOR

This paper is the first part of a study of profinite completion and profinite spaces from a functorial point of view. The study aims at understanding mapping spaces, classifying spaces, and other notions involving higher homotopies in the setting of profinite spaces.

The term *rigid profinite space* will apply to a left filtered (see below) actually commuting diagram of spaces with finite homotopy groups, as distinguished from an Artin-Mazur profinite space which is a homotopy commuting diagram. One advantage in working with actually commuting diagrams is that a *functorial homotopy limit* exists relating diagrams of spaces to spaces. A homotopy limit does not exist in general for diagrams in the homotopy category. The actual relationship between rigid profinite spaces and profinite spaces is not well understood. Theorem 3.4 and Corollary 6.9 below suggest that the relationship is very close. Rigid profinite spaces arise naturally in many contexts; in particular, the étale homotopy type of a nice variety may be made rigid [8], [10].

The main results of this paper are as follows. In § 2 we construct a *functorial profinite completion* $\hat{X}$ for a connected space X taking values in the category of rigid profinite spaces. We employ a construction pioneered by Quillen [13] making use of the profinite completion of a simplicial group. We show that this profinite completion is weakly equivalent to that of Artin-Mazur. A functorial nilpotent p-completion has been studied extensively by Bousfield and Kan [5]; it gives equivalent results on spaces X which are nilpotent and *locally of finite type* — i.e., $H^*(X; M)$ of finite type for all finite local coefficient systems M. For spaces which are not nilpotent, the Bousfield-Kan completion gives very different results.

In § 3 we construct a *functorial discretization* dY for a rigid profinite space Y. We show that *for X a connected space, $d\hat{X}$ represents the Sullivan finite completion of X* [17] in a strong sense — i.e., for all connected spaces Z, the mapping space hom$(Z, d\hat{X})$ has the right higher homotopy groups (see Theorem 3.4). In part II of this paper (in preparation), we will show that when X is locally of finite type, $\hat{X}$ and dY are *homotopy adjoint* in a strong sense — i.e., hom (X, dY) is weakly equivalent to hom $(\hat{X}, Y)$ for a natural definition of hom $(\hat{X}, Y)$.