PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 1, 1980

RIGHT DERIVATIONS AND RIGHT DIFFERENTIAL OPERATORS

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Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

In the beginning are defined right differential operators for noncommutative algebras. Differential operators give rise to right derivations, jet and Kähler modules for noncommutative algebras, differential separability and differential inseparability (called differentially stacked). Differential separability is related to the abelianized algebra being separable, Differentially stacked is related to pure inseparability of noncommutative algebras.

Introduction. Suppose A is a not necessarily commutative algebra with right A-modules M, N. For $f \in \mathrm{Hom}(M, N)$, $a \in A$, $m \in M$ let $[f, a] \in \mathrm{Hom}(M, N)$ be defined by $[f, a](m) \equiv f(ma) - f(m)a$. So $\mathrm{Hom}_A(M, N) = \{f \in \mathrm{Hom}(M, N) | [f, a] = 0$ for all $a \in A\}$. Inductively define $\mathscr{D}_A^0(M, N) = \mathrm{Hom}_A(M, N)$ (or $\mathscr{D}_A^{-1}(M, N) = \{0\}$) and

$$\mathscr{D}_A^n(M, N) = \{f \in \mathrm{Hom}(M, N) | [f, a] \in \mathscr{D}_A^{n-1}(M, N) \quad \text{for all} \quad a \in A\} .$$

$\mathscr{D}_A^n(M, N)$ is the module of nth order right differential operators from M to N. This follows the definition in [3, (2.1.1) p. 210] where A was assumed commutative.

An important example of first order right differential operator is the *right derivation*, a linear map $d: A \to N$ satisfying $d(a\alpha) = d(a)\alpha + d(\alpha)a$. $\mathscr{D}_A^1(A, N)$ splits up as a direct sum of $\mathrm{Hom}_A(A, N) = N$ and the module of right derivations from A to N. More about right derivations in a while.

As in the commutative case $\mathscr{D}_A^n(M, -)$ has a universal representing object. A right A-module $J_n(M)$ together with $j_n \in \mathscr{D}_A^n(M, J_n(M))$ where for each $f \in \mathscr{D}_A^n(M, N)$ there is a unique $F \in \mathrm{Hom}_A(J_n(M), N)$ with $Fj_n = f$. The key to dropping the commutativity requirement on A is the introduction of the opposite algebra $\bar{A}$ and the enveloping algebra $A^e = \bar{A} \otimes A$. A fair amount of our first section is an adaptation of [3, §2, p. 210-220] to the noncommutative, a process of discovering which A's to leave and which must become $\bar{A}$'s.

The (*jet*) modules $J_n(M)$ depend on $Z_n(A)$—an A-bimodule—in that $J_n(M) \cong M \otimes_A J_n(A)$ an isomorphism preserving universal properties. A is *right differentially separable* if for all n (or $n = 2$) $(j_n, J_n(A)) \sim (I, A)$. This is equivalent to $\mathscr{D}_A^n(M, N) = \mathrm{Hom}_A(M, N)$ for all n and right A-modules M, N. In other words there are no interesting (non A-linear) right differential operators. The opposite