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# DIFFERENTIAL VALUATIONS

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*Dedicated to Gerhard Hochschild on the occasion of his 65th birthday*

**A notion of "differential valuation" is defined for ordinary differential fields of characteristic zero by postulating for a given valuation of the field a natural analogue of the elementary L'Hospital's rule. Such valuations occur implicitly in classical analysis, for example in Hardy's orders of infinity and in the study of singular points of systems of ordinary differential equations. The fundamental properties of differential valuations are worked out in this paper, numerous examples are discussed, and it is shown that a differential valuation can always be extended to an algebraic extension field. Applications are anticipated to the study of singularities of algebraic differential equations.**

In the following, $k$ will denote an ordinary differential field of characteristic zero, $'$ its derivation, and $C$ its subfield of constants.

THEOREM 1. *Let $k$ be a differential field of characteristic zero with subfield of constants $C$, $v$ a valuation of $k$ that induces the trivial valuation on $C$, and let $\mathfrak{o}$ and $\mathfrak{m}$ be respectively the valuation ring of $v$ and its maximal ideal. Then the following statements are equivalent*:

(1) *If $a \in \mathfrak{o}$, $b \in \mathfrak{m}$, $b \neq 0$, then $a'b/b' \in \mathfrak{m}$.*

(2) *If $a, b \in k^*$ and $0 < v(a) \leqq v(b)$, then $(b/a - b'/a') \in \mathfrak{m}$.*

(3) *If $a, b \in k^*$ and $v(a) \leqq v(b) < 0$, then $(b/a - b'/a') \in \mathfrak{m}$.*

(4) *If $a \in \mathfrak{o}$, $b \in k^*$, $1/b \in \mathfrak{m}$, then $a'b/b' \in \mathfrak{m}$.*

First note that $b \in \mathfrak{m}$ if and only if $v(b) > 0$. Also, $C \cap \mathfrak{m} = \{0\}$, so that if $b \in \mathfrak{m}$ and $b \neq 0$, then $b' \neq 0$.

*Proof that* (1) $\Rightarrow$ (2). Under the assumptions of (2) we can write $b = ac$, with $c \in \mathfrak{o}$. Since $a \in \mathfrak{m}$ we get

$$\frac{b}{a} - \frac{b'}{a'} = c - \frac{a'c + ac'}{a'} = -\frac{c'a}{a'},$$

and by (1) this last quantity is is $\mathfrak{m}$.

*Proof that* (2) $\Rightarrow$ (3). Under the assumptions of (3), if we write $\alpha = 1/a$, $\beta = 1/b$, we get $0 < v(\beta) \leqq v(\alpha)$, so that $0 < v(\alpha) \leqq v(\alpha^2/\beta)$. Hence

$$\frac{b}{a} - \frac{b'}{a'} = \frac{(1/\beta)}{(1/\alpha)} - \frac{(1/\beta)'}{(1/\alpha)'} = \frac{\alpha}{\beta} - \frac{\alpha^2\beta'}{\beta^2\alpha'} = \frac{(\alpha^2/\beta)'}{\alpha'} - \frac{\alpha^2/\beta}{\alpha},$$