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ON UNIVERSAL EXTENSIONS OF DIFFERENTIAL FIELDS

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Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

The main result of this paper is the following:

THEOREM: **Let $\mathscr{U}$ be a universal extension of the differential field $\mathscr{F}$ of characteristic zero and let $\mathscr{G}$ be a strongly normal extension of $\mathscr{F}$ in $\mathscr{U}$. Then $\mathscr{U}$ is a universal extension of $\mathscr{G}$.**

Introduction. We deal with differential fields, always of characteristic zero, relative to a nonempty finite set of commuting derivation operators. By an *extension* of a differential field, we always mean a differential field extension. An extension $\mathscr{F}'$ of a differential field $\mathscr{F}$ is said to be *finitely generated* if $\mathscr{F}'$ has a finite subset Φ such that $\mathscr{F}' = \mathscr{F}\langle\Phi\rangle =$ the smallest extension of $\mathscr{F}$ in $\mathscr{F}'$ that contains Φ.

Let $\mathscr{F}$ be a differential field. Recall that an extension $\mathscr{U}$ of $\mathscr{F}$ is called *universal* if, for any finitely generated extension $\mathscr{F}_1$ of $\mathscr{F}$ in $\mathscr{U}$ and any finitely generated extension $\mathscr{G}$ of $\mathscr{F}_1$ not necessarily in $\mathscr{U}$, $\mathscr{G}$ can be embedded in $\mathscr{U}$ over $\mathscr{F}_1$, i.e., there exists an extension of $\mathscr{F}_1$ in $\mathscr{U}$ that is isomorphic (in the sense of differential fields) to $\mathscr{G}$ over $\mathscr{F}_1$. Such a universal extension of $\mathscr{F}$ always exists ([2] p. 132, Th. 2). It is not unique, but if $\mathscr{U}$ and $\mathscr{V}$ are two universal extensions of $\mathscr{F}$, then there exist universal extensions $\mathscr{U}'$ and $\mathscr{V}'$ of $\mathscr{F}$, lying in $\mathscr{U}$ and $\mathscr{V}$, respectively, such that $\mathscr{U}'$ is isomorphic to $\mathscr{V}'$ over $\mathscr{F}$ ([2] p. 135, Exerc. 7).

Let $\mathscr{U}$ be a universal extension of the differential field $\mathscr{F}$ and let $\mathscr{G}$ be an extension of $\mathscr{F}$ in $\mathscr{U}$. Under favorable conditions, $\mathscr{U}$ is then a universal extension of $\mathscr{G}$, too. For example, this is the case when $\mathscr{G}$ is finitely generated over $\mathscr{F}$ ([2] p. 133, Prop. 4), and also when $\mathscr{G}$ is algebraic over $\mathscr{F}$ ([2] p. 134, Exerc. 1). The main purpose of the present note is to point out another such favorable condition. We shall show (§1) that when $\mathscr{G}$ is a strongly normal extension of $\mathscr{F}$, in the general sense of Kovacic [4] (i.e., not necessarily finitely generated), then $\mathscr{U}$ is universal over $\mathscr{G}$. This result shows that, in the study of strongly normal extensions, it is not necessary to replace $\mathscr{U}$ by a larger universal extension of $\mathscr{F}$ (see Kovacic [4] p. 518).

Every strongly normal extension of $\mathscr{F}$ in $\mathscr{U}$ is embeddable over $\mathscr{F}$ in a constrained closure of $\mathscr{F}$ in $\mathscr{U}$ ([3] p. 162, Th. 3 or Blum [1] p. 42 (15)) and hence, in particular, is constrained over $\mathscr{F}$