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GROUPS OF INTEGRAL REPRESENTATION TYPE

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Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

Introduction. Let Γ be a group, $\rho: \Gamma \rightarrow GL_n(K)$ a matrix representation over some field K, and χ_ρ its character: $\chi_\rho(s) = \mathrm{Tr}(\rho(s))$. The theme of this paper is, generally spaking, to draw conclusions about Γ or $\rho(\Gamma)$ from finiteness assumptions on $\chi_\rho(\Gamma)$. The prototype of such results is Burnside's theorem saying, when ρ is absolutely irreducible, that if $\chi_\rho(\Gamma)$ is finite then $\rho(\Gamma)$ is finite. This yielded his affirmative solution of the "Burnside problem" for linear groups. The same argument shows, when K is a locally compact field (like $\boldsymbol{R}$ or $\boldsymbol{C}$) that we may replace "finite" by "bounded", and conclude from boundedness of $\chi_\rho(\Gamma)$ that the closure of $\rho(\Gamma)$ is compact. This yields an affirmative answer to a question posed to us by Ken Millett. Independently, Kaplansky asked us whether a subgroup of $GL_n(\boldsymbol{C})$, each element of which is conjugate to a unitary matrix, is itself conjugate to a subgroup of $U_n(\boldsymbol{C})$. We give a counterexample. These results occupy § 1.

The rest of the paper is devoted to the introduction and study of the following notion. Let A be a commutative ring and n an integer $\geqq 1$. A group Γ is said to have *integral n-representation type* over A if, for any field K which is an A-algebra, and any representation $\rho: \Gamma \rightarrow GL_n(K)$, the elements $\chi_\rho(s) \in K$ for $s \in \Gamma$ are all integral over A, i.e., roots of a monic polynomial with coefficients in A (Def. 5.1). We conclude from this, when A is noetherian, that, for any finite subset X of Γ, the sub A-algebra of $M_n(K)$ generated by $\rho(X)$ is a finitely generated A-module, (Prop. 5.2). Further, Γ has only finitely many conjugacy classes of irreducible representations of dimension $\leqq n$ over any field K as above (Prop. 5.3). These and other strong finiteness properties are deduced from the theory of rings with polynomial identities, as developed in Procesi's book [12]. In §§ 2-4 we give a rendering of this source material adapted to the present applications.

The case of main interest is when $A = Z$, which we now assume. A group Γ has integral 1-representation type if and only if Γ^{ab} is a torsion group (Prop. 5.5). Serre [15] has furnished a class of finitely generated groups Γ of integral 2-representation type, namely those with the fixed point property for actions on trees (Th. 6.4). This is equivalent to Γ^{ab} being finite and Γ not being a nontrivial amalgamated free product (Th. 6.2). We derive a useful refinement (Th. 6.5) of Serre's theorem in order to prove (Cor. 6.7) a conjecture