PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 2, 1980

TCHEBYCHEFF SYSTEMS AND BEST PARTIAL BASES

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This is a contribution to the *partial basis problem* and, in particular, to the case where the basis elements are cosigns or consecutive powers cosines. We contribute also to the general theory of Tchebycheff systems to which the partial basis problem is strongly related.

1. Introduction. The *partial basis problem* was formulated and studied by J. T. Lewis, D. W. Tufts and the author in 1975 in connection with their study of optimization of multichannel processing. Let X be a normed linear space, let $f, h_1, h_2, \cdots, h_N \in X$ and let n be an integer, $1 \leqq n < N$. For every sequence $\mu = \{\mu_k\}_1^n$ of integers, with $1 \leqq \mu_1 < \mu_2 \cdots < \mu_n \leqq N$, consider

$$e(\mu) = \min\left\|f - \sum_{k=1}^{n} c_k h_{\mu_k}\right\|$$

where the minimum is taken over all possible choices of the scalars $c_1, \cdots, c_n$. The problem is to minimize $e(\mu)$. It is of particular interest when X is one of the standard function spaces.

Subsequently, progress has been made both in theory and in the computational aspect. An algorithm, numerical examples and some theoretical results have been given by K. M. Levasseur and J. T. Lewis in [6]. G.G. Lorentz [5] has observed that, for $X = L^2(0, 1)$, h_k the function $x^{k-1}(k = 1, 2, \cdots, N)$ and f the function x^N, $e(\mu)$ is minimized by $\mu = \{N - n + 1, N - n + 2, \cdots, N\}$ and conjectured the same to be true for $X = C[0, 1]$. This was proved by I. Borosh, C. K. Chui and P. W. Smith [1, Theorem 1].

In Theorem 4 below we give a sufficient condition for a real function f, continuous on $[a, b](0 < a < b < \infty)$, that $e(\mu)$ be minimized (only) by $\mu = \{N - n + 1, N - n + 2, \cdots, N\}$, where $X = L^p(a, b)$, $1 \leqq p \leqq \infty$ and h_k is the function $x^{k-1}(k = 1, 2, \cdots, N)$. For such a function f (with $a = 0, b = \pi$), Theorem 17 gives such a sufficient condition with the same X, where each h_k is a function of the form $\cos \alpha x$.

In proving Theorems 4 and 17 we use Theorem 1 which, together with Lemma 2, is due to A. Pinkus. The author is very grateful to him for that as well as for other valuable remarks. Cf. also [8, § 3].

The partial basis problem turns out to be very much interrelated with the theory of Tchebycheff systems and this paper is a contribution to both. Thus in Theorem 8 we characterize certain tri-