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AN ALGEBRAIC EXTENSION OF THE LAX-MILGRAM THEOREM

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In this work a Lax-Milgram type theorem is proved for quadratic spaces over a division ring K with involution $*$, say, whose center contains an ordered domain P such that for every element a in K, $aa^* = |a|^2$, (where $|a|$, the absolute value of a, is in P^+ which is the set of positive elements of P), and for every element b in P^+ there exists an element c in P^+, denoted by $b^{1/2}$, such that $c^2 = b$. Specifically, with the above assumptions on K, the following is proved:

Let (H_i, Φ_i) $i = 1, 2$ be quadratic spaces over K such that for each u in H_2 $\sup|\Phi_2(u, v)|(|\Phi_2(v, v)|^{1/2})^{-1}$ exists and equals $|\Phi_2(u, u)|^{1/2}$. Let $B: H_1 \times H_2 \to K$ be an orthocontinuous bilinear form satisfying:

(i) $\inf_{x\neq 0} \sup_{y\neq 0} |B(x, y)|(|\Phi_1(x, x)|^{1/2}\,|\Phi_2(y, y)|^{1/2})^{-1}=\gamma$ exists and $\gamma - \delta$ is in P^+ for some δ in P^+.

(ii) $\sup|B(x, y)|$ exists and is in P^+ for all $y \neq 0$ $x \in H$.

Then given any orthocontinuous linear functional ϕ on H_2 whose kernel is splitting there exists a unique element x_0 in H_1 such that $\phi(y) = B(x_0, y)$ for all y in H_2.

Moreover

$$\delta^{-1} \sup_{y\neq 0} |\phi(y)|\,(|\Phi_2(y, y)|^{1/2})^{-1} - |\Phi_1(x_0, x_0)|^{1/2} \in P^+ \cup \{0\}\,.$$

1. Introduction. Motivated by what he referred to as "happy accidents in the Hilbert space theory that correlate algebraic and topological considerations" Piziak [3] proposed an algebraic approach to the study of sesquilinear forms in infinite dimensions. In this approach he introduced the notion of quadratic spaces by the means of which he obtained an algebraic generalization of some Hilbert space results. He proved an algebraic version of the Riesz-Frechet Representation Theorem and discussed continuity all in the algebraic context of a vector space over a division ring in which no natural topology is present.

We here consider an algebraic extension of the Lax-Milgram Theorem, a variant of the Riesz Representation Theorem.

Our results are of pure algebra. We have not assumed a topology either on the division ring or on the vector space which we considered. It is thus interesting to note that these results imply their analogous standard topological results in the context of Hilbert space.