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GOOD CHAINS WITH BAD CONTRACTIONS

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Let $R \subset T$ be commutative rings with T integral over R. In the study of chains of prime ideals, it is often of interest to know about primes $q \subset q'$ of T such that height $(q'/q) <$ height $(q' \cap R/q \cap R)$. In this paper we will consider a chain of primes $q_1 \subset q_2 \subset \cdots \subset q_m$ in T which is well behaved in that height $(q_m/q_1) = \sum_{i=2}^{m}$ height (q_i/q_{i-1}), but which suffers the pathology that height $(q_i \cap R/q_{i-1} \cap R) >$ height (q_i/q_{i-1}) for each $i = 2, \cdots, m$. Our goal is to find a bound on how large m can be.

Our main result is that if T is generated as an R-module by n elements, then there is a bound b_n such that $m \leqq b_n$; moreover $b_2 = 2$ and in general $b_n \leqq b_{n-1}^{n-2} + b_{n-1}^{n-3} + \cdots + b_{n-1} + 2$. Let us quickly add that we do not claim that this formula gives the best bound possible. (We rather suspect not.) If $c = b_{n-1} + 2$, we also have, as part of our main result, that $m \leqq$ height $(q_c/q_1) + b_{n-1}$. (If $m > b_{n-1}$, so that q_c exists.) Finally, if we have the added assumption that height $(q_i/q_{i-1}) \leqq r$ for $i = 2, \cdots, m$, then $m \leqq 2(r+1)^{n-2}$.

The bulk of our effort is needed to discuss the case that $T = R[u]$ is a simple integral extension of R. This is done in § 3. That section also introduces a new "going down" technique of some interest. Section 2 treats a highly special situation in which we obtain a much sharper bound. This case has some interest in its own right and also starts an induction needed in § 3. The fourth section gives the main result mentioned above. Lastly, in § 5, we present some examples. These illustrate the point that there is no bound in general, even in the case of Noetherian domains, on m which is independent of the size of the integral extension $R \subset T$. Specifically, we show that $b_n \to \infty$ as $n \to \infty$. Thus our bounds, while presumably not sharp, have the proper form.

DEFINITION. The chain of primes $P_1 \subset P_2 \subset \cdots \subset P_m$ is *taut* if height $(P_m/P_1) = \sum_{i=2}^{m}$ height (P_i/P_{i-1}).

NOTATION. The following notation will be standard throughout except when specifically indicated otherwise. $R \subset T$ will be an integral extension of domains, $q_1 \subset \cdots \subset q_m$ will be a taut chain of primes in T lying over $p_1 \subset \cdots \subset p_m$ in R. Height(p_m/p_1) will be finite and height$(p_i/p_{i-1}) >$ height(q_i/q_{i-1}), $i = 2, \cdots, m$. Finally, x will be an indeterminate.