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BIHARMONIC AND POLYHARMONIC PRINCIPAL FUNCTIONS

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The biharmonic principal function problem is the construction of a biharmonic function in a space which "imitates" the behavior of a given singularity function. In this paper we first define the notion of a biharmonic operator which clarifies the modes of "imitation." We then prove the existence and uniqueness theorem of the biharmonic principal function. The theory is a generalization of the harmonic principal functions to the larger family of biharmonic functions. An indication of its application as well as its further generalization to polyharmonic functions is also given.

The theory of principal functions plays an important role in the study of harmonic functions in that it allows for the global construction of harmonic functions with a great variety of singularity behaviors. See Ahlfors-Sario [1] or Rodin-Sario [3] for a comprehensive treatment of this theory and many of its applications. Since the study of biharmonic and polyharmonic functions draws heavily from the experience of harmonic theory, it has been felt that a theory of biharmonic or polyharmonic principal functions would be desirable. Except for some results in the thesis of Rader [2] which constructs some interesting special examples, a general theory is still in the waiting. In this paper, we will present another step toward such a theory in first defining the basic concept of biharmonic operator and then proving the existence and uniqueness of biharmonic principal functions, one of the three main theorems of the theory. Our paper is self contained except for a proof of Sario's q-lemma which can be found in [1] or [3]. An indication of generalizing the results to polyharmonic functions is given at the end.

In § 1, after a review of notations, we prove an alternating lemma which is the main technical tool of our theory. From this, the main existence and uniqueness theorem of Sario et al for the harmonic theory follows rather easily.

In § 2, we define the concepts of a biharmonic operator which is basic for our theory. Examples and simple properties of this operator are given. Next in § 3, we prove the existence and uniqueness theorem for biharmonic principal functions. Some applications are then given in § 4 which includes the construction of biharmonic functions with various singularity properties, e.g., the classical singularity problem of closed manifolds, the biharmonic