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ONE-PARAMETER SEMIGROUPS OF ISOMETRIES INTO H^p

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In this paper we explicitly describe all strongly continuous one-parameter semigroups $\{T_t\}$ of isometries of $H^p(D)$ into $H^p(D)$, where $1 \leqq p < \infty$, $p \neq 2$, and D is the unit disc $|z|<1$ in the complex plane C. It turns out (Theorem (1.6)) that for each t, $T_t = \psi_t U_t$, where U_t is a surjective isometry and ψ_t is an inner function (the families $\{\psi_t\}$ and $\{U_t\}$ are uniquely determined provided $\{U_t\}$ is suitably normalized). The nature of the family $\{\psi_t\}$ depends on the set of common fixed points of the family of Möbius transformations of the disc associated with the family $\{U_t\}$. If there is exactly one common fixed point in D, then $\{T_t\}$ must consist of surjective isometries (§ 4); otherwise $\{T_t\}$ consists of surjective isometries only in very special cases (§§ 2, 5). The families $\{\psi_t\}$ are explicitly described in this paper.

1. Preliminaries. The linear isometries of H^p into H^p were characterized by Forelli [7, Theorem 1]. For convenience we quote here a part of the statement of Forelli's theorem.

THEOREM. *Let T be a linear isometry of H^p into H^p, $1 \leqq p < \infty$, $p \neq 2$. Then T has a unique representation*

$$Tf = Ff(\phi), \text{ for all } f \in H^p , \tag{1.1}$$

where F is analytic on D, and ϕ is a nonconstant inner function.

Let $\boldsymbol{R}$ be the set of real numbers, and $\boldsymbol{R}^+$ be $\{t \in \boldsymbol{R}: t \geqq 0\}$. Let $\{T_t\}$, $t \in \boldsymbol{R}^+$, be a strongly continuous one-parameter semigroup of isometries of H^p into H^p, $1 \leqq p < \infty$, $p \neq 2$. For each $t \in \boldsymbol{R}^+$, let F_t and ϕ_t be as in the representation (1.1) for T_t. From the identity $T_{s+t} = T_s T_t$ we get for all $s, t \in \boldsymbol{R}^+$:

$$\phi_{s+t} = \phi_s \circ \phi_t \tag{1.2}$$

$$F_{s+t} = F_s F_t(\phi_s) , \tag{1.3}$$

where "$\circ$" denotes composition of maps. Let Z be the identity map, $Z(z) = z$. Obviously $F_t = T_t 1$, and $T_t Z = F_t \phi_t$. It follows by strong continuity that if $u \in \boldsymbol{R}^+$, $z_0 \in D$, and $F_u(z_0) \neq 0$, then $\phi_t(z_0) \to \phi_u(z_0)$ as $t \to u$. From this and the fact that $\{\phi_t: t \in \boldsymbol{R}^+\}$ is normal, we find that $t \mapsto \phi_t$ is continuous from $\boldsymbol{R}^+$ to the usual metric space of all analytic functions on D. Using this and the pointwise equicontinuity of $\{\phi_t: t \in \boldsymbol{R}^+\}$, we infer that $\phi_t(z)$ is a continuous function of (t, z)