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THE APPROXIMATION OF UPPER SEMICONTINUOUS MULTIFUNCTIONS BY STEP MULTIFUNCTIONS

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Let P be a right rectangular parallelepiped in R^m and let Y be a metric space. If $\Gamma: P \to Y$ is an upper semicontinuous multifunction such that for each x in P the set $\Gamma(x)$ is nonempty and closed, then there exists a sequence $\{\Gamma_k\}$ of upper semicontinuous closed valued step multifunctions convergent in terms of Hausdorff distance to Γ from above. If Γ is compact valued and increasing and P is a closed interval, then the convergence can be made uniform. As a consequence of a Dini-type theorem for mutifunctions, the convergence can also be made uniform if Γ is compact valued and continuous.

1. Introduction. Let X and Y be topological spaces. A *multifunction* $\Gamma: X \to 2^Y$ assigns to each x in X a subset $\Gamma(x)$ of Y, possibly empty. A multifunction Γ is called *upper semicontinuous* (u.s.c.) at z in X if whenever V is an open subset of Y that contains $\Gamma(z)$ then the set $\{x: \Gamma(x) \subset V\}$ contains a neighborhood of z. It is called *lower semicontinuous* (l.s.c.) at z if whenever an open subset V of Y satisfies $V \cap \Gamma(z) \neq \emptyset$, then $\{x: \Gamma(x) \cap V \neq \emptyset\}$ contains a neighborhood of z. It is called *continuous* at z if Γ is both u.s.c. and l.s.c. at z, and Γ is continuous (resp. u.s.c., l.s.c.) in X if Γ is continuous (resp. u.s.c., l.s.c.) at each point of X.

The basic theory of semicontinuous multifunctions is presented in Berge [3], Kuratowski [6], and Smithson [9]. We now list a few tangible semicontinuous multifunctions, the first of which is mentioned in [6].

EXAMPLE 1. If $f: Y \to X$ is onto, then $f^{-1}: X \to 2^Y$ is an u.s.c. (resp. l.s.c.) multifunction if and only if f is a closed (resp. open) single valued function.

EXAMPLE 2. If C is a closed convex set in m-dimensional Euclidean space R^m, then $\Gamma: C \to 2^{R^m}$ defined by

$$\Gamma(z) = \{y: ||y|| \leqq 1 \text{ and } y \cdot (x - z) \leqq 0 \text{ for each } x \text{ in } C\}$$

is an u.s.c. multifunction.

EXAMPLE 3. If C is an arbitrary set in R^m, then $\Sigma: C \to 2^{R^m}$ defined by