PACIFIC JOURNAL OF MATHEMATICS
Vol. 87, No. 2, 1980

UNIQUE BEST APPROXIMATION FROM A C^2-MANIFOLD IN HILBERT SPACE

THEAGENIS ABATZOGLOU

Given a C^2-manifold in a Hilbert space we examine whether a critical point of the distance function to the manifold is actually a global best approximation. We establish a criterium for the above in terms of the curvature of the manifold.

1. Introduction. In the last 10 years there have been papers, see [3], [4], [9], [10], [12], where the authors determine how close you have to be from a manifold to make sure that a critical point of the distance function to the manifold is a global best approximation. In their discussion the authors above use explicitly or implicitly the notions of curvature and radius of curvature and as long as the manifold does not bend back into itself too much, they use a global lower bound on the radius of curvature to conclude that if a point is within one third of the radius of curvature then it has a unique global approximation. In this paper we use different methods, mainly from differential geometry, to arrive at a sharp bound of one radius of curvature guaranteeing unique global best approximation.

2. Global best approximation to C^2-curves. We would like to establish some facts about C^2-curves which we will use later to obtain our results about global best approximation to C^2-manifolds. A version of Theorem 2.1b is known in n-dimensional Euclidean space and is due to Schwarz, see [7] page 38. Our proof holds for any Hilbert space and is different from the classical one.

We will need the following lemma to prove Theorem 2.1:

LEMMA 2.1. *Let x, y, z be in H, a Hilbert space, and assume that $\|y\| = 1$, $y \perp z$. Then:* $|(x, z)| \leqq \|z\| \sqrt{\|x\|^2 - (x, y)^2}$.

Proof. Write z as: $z = t(x - (x, y)y) + v$ where $v \perp x, y$. Then $\|z\|^2 = t^2(\|x\|^2 - (x, y)^2) + \|v\|^2$ so that: $|t| \leqq \|z\| / \sqrt{\|x\|^2 - (x, y)^2}$. We now estimate (x, z) and get

$$|(x, z)| = |t\,\|x\|^2 - t(x, y)^2| = |t(\|x\|^2 - (x, y)^2)|$$
$$\leqq \|z\| \sqrt{\|x\|^2 - (x, y)^2}\,.$$

THEOREM 2.1. *If γ is a C^2-curve embedded in H such that*