PACIFIC JOURNAL OF MATHEMATICS
Vol. 88, No. 1, 1980

RATIONAL FUNCTIONS WITH POSITIVE COEFFICIENTS, POLYNOMIALS AND UNIFORM APPROXIMATIONS

PETER B. BORWEIN

Upper bounds are established for the uniform approximation of continuous functions on [1, 0] by rational functions with positive coefficients. These bounds are obtained by rewriting polynomials with no positive roots as rational functions with positive coefficients.

1. Introduction. The uniform closure in $C[1, 0]$ of the set of polynomials with positive coefficients includes only those functions analytic in the unit disc whose power series expansions have nonnegative coefficients. The uniform closure of the set of rational functions with positive coefficients consists of all continuous functions which are never negative on [0, 1]. This is a consequence of the following interesting factorization theorem.

THEOREM 1. (*E. Meissner* [3].) *Suppose that p is a polynomial with real coefficients and that $p(x) > 0$ for $x > 0$. Then there exists a rational function $r(x)$ with nonnegative coefficients so that $p(x) = r(x)$.*

We will provide an accurate bound for the degree of the above r in terms of the degree of p and some knowledge of the location of the roots of p. We will also derive some estimates concerning how efficiently polynomials can be approximated on [0, 1] by rational functions with positive coefficients. We will exploit these results to prove a number of approximation theorems. For instance: if f is analytic in some neighborhood of [0, 1] and positive on [0, 1], then there exists a sequence of rational functions $\{r_n\}$ where each r_n is of degree n and has nonnegative coefficients so that $||f-r_n||_{[0,1]} = 0(\alpha^{-\sqrt{n}})$ for some $\alpha > 1$.

We employ the following notation. Let $\prod_n$ denote the polynomials with real coefficients of degree at most n. Let $\prod_n^+$ be the sub class of $\prod_n$ whose elements have nonnegative coefficients. Let R_n^{++} denote those rational functions p_n/q_n where $p_n, q_n \in \prod_n^+$. For $f \in C[a, b]$ define

$$\prod_n(f\colon [a, b]) = \inf_{p \in \Pi_n} ||f - p||_{[a,b]}$$

$$\prod_n^+(f\colon [a, b]) = \inf_{p \in \Pi^+} ||f - p||_{[a,b]}$$

$$R_n^{++}(f\colon [a, b]) = \inf_{r \in R^{++}} ||f - r||_{[a,b]}$$