PACIFIC JOURNAL OF MATHEMATICS
Vol. 88, No. 1, 1980

# MONOIDAL CLOSED, CARTESIAN CLOSED AND CONVENIENT CATEGORIES OF TOPOLOGICAL SPACES

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**This paper develops a unified theory of function spaces $M_{\mathscr{A}}(Y, Z)$ with set-open topologies, the sets in question being the continuous images of selected classes of topological spaces $\mathscr{A}$. We prove that at least five of these function spaces are distinct and have corresponding exponential homeomorphisms $\theta: M_{\mathscr{A}}(X, M_{\mathscr{A}}(Y, Z)) \cong M_{\mathscr{A}}(X \times_{\mathscr{A}} Y, Z)$ for suitably retopologized product spaces $X \times_{\mathscr{A}} Y$. Singleton spaces are normally identities with respect to these products and so we have determined four distinct monoidal closed structures for the category of all spaces. Conditions for the category of spaces generated by $\mathscr{A}$, i.e., the coreflective hull of $\mathscr{A}$, to be cartesian closed and/or convenient are given. One result asserts that the category of sequential spaces is the smallest convenient category.**

R. Brown proved in [4, p. 240, Corollary 1.8] that there is an exponential homeomorphism for the category HAUS of all Hausdorff spaces, relating function spaces with the compact-open topology to a retopologized product $X \times_s Y$ without any further restrictions on $X$ and $Y$. This product has the weak topology with respect to all subspaces of the usual product $X \times Y$ of the form:

$$X \times B \text{ where } B \text{ is compact in } Y, \text{ and } \{x\} \times Y \text{ where } x \in X .$$

It is also shown in the same paper [4, p. 242, Remark 1.15] that there is a similar general exponential law for TOP relating function spaces with the topology of pointwise convergence to another suitably retopologized product. It is known [24, p. 277] that an analogous result holds for function spaces with the indiscrete topology. This is not a completely trivial example as the product used does not have the discrete topology, see example (ii) in §8 below. Hence we know of two general exponential laws for TOP and, by restriction, three such for HAUS. Wyler [25, p. 227] raises the question of finding new closed structures for TOP. Another similar theory is developed in [21, Chapter 5] for function spaces with the $cs$-open (convergent sequence open) topology of [12, 13]. It is shown ([21, p. 61] and example (i) §6 below) that the corresponding $\theta$ is a continuous bijection; we have not been able to determine if it is, in general, a homeomorphism.