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MAPS ON SIMPLE ALGEBRAS PRESERVING ZERO PRODUCTS. I: THE ASSOCIATIVE CASE

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Recent studies of linear transformations of various types on the space of $n \times n$ matrices over a field suggest the general problem of finding the semilinear transformations f on an algebra A over a field k, with the property that

$$xy = 0 \Rightarrow f(x)f(y) = 0 ,$$

where $x, y \in A$. In this article such maps are determined for a class of primitive associative algebras, including the case of bijective maps f on a finite-dimensional simple associative algebra A.

Introduction. In recent years there have been many investigations characterizing the linear transformations on the $n \times n$ matrix algebra $M_n(k)$ over a field k which preserve various properties, one of the earliest being a theorem of Dieudonné finding the bijective linear maps preserving the set of singular matrices in $M_n(k)$ [4], with other studies concerning maps preserving rank, various algebraic groups, etc. [1, 2, 5, 6, 8]. In many cases, the problems considered can be easily formulated entirely in terms of the structure of $M_n(k)$ as an associative or Lie algebra, and can therefore be investigated in other algebras. For example, a theorem of Watkins [9] finds the bijective linear transformations f on $M_n(k)$ satisfying the condition that $[f(x), f(y)]=0$ for all pairs of elements x, y of $M_n(k)$ such that $[x, y] = 0$, where $[\ ,\ ]$ denotes the usual Lie product in $M_n(k)$. This naturally suggests that the same problem be investigated for other algebras. If A is any algebra (not necessarily an associative or Lie algebra) over a field k, we can seek to determine the set $G(A)$ of all bijective linear transformations f on A with the property that $f(x)f(y) = 0$ for all pairs of elements x, y of A such that $xy = 0$. We say that such a map f *preserves zero products*.

If A is finite-dimensional, the set of all ordered pairs x, y for which $xy = 0$ is an algebraic set in the affine space $A \times A$, and an easy modification of the proof of a lemma of Dixon [5, p. 386] shows that $G(A)$ is a group. Clearly $G(A)$ contains the automorphism group G_1 of A, the group of units G_2 of the centroid of A (the algebra of linear transformations which commute with both left and right multiplications in A), and the group G_3 of all bijective transformations f of the form $f(x) = x + g(x)$, where g is a linear map of A into the ideal A^0 of all elements z for which $zA = Az = 0$. The product $G_0(A) = G_1G_2G_3$ is a subgroup of $G(A)$. (If $A^0 = 0$,