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# ARCHIMEDEAN LATTICE-ORDERED FIELDS THAT ARE ALGEBRAIC OVER THEIR $o$-SUBFIELDS

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**Several properties of archimedean lattice-ordered fields which are algebraic over their $o$-subfield will be shown to be equivalent. Among these properties are the following: Two geometric descriptions of the positive cone. A sufficient condition for an intermediate field of the lattice-ordered field and its $o$-subfield to be lattice-ordered. A description of the additive structure of the lattice-ordered field. Two statements on the extendibility of lattice orders to total orders. A statement on the extendibility of a given lattice order to a lattice order on a real closure.**

Introduction. It has been shown in [4], Kap. 2 that each archimedean $l$-field (=lattice-ordered field) $K$ with positive cone $P_K$ has a largest subfield $L$ which admits a total order $P_L$ with $P_L \cdot P_K \subseteq P_K$. $L$ is called the $o$-subfield of $K$. In this paper archimedean $l$-fields that are algebraic over their $o$-subfield will be investigated. In §1 several geometric and structural properties of $l$-fields are considered. §2 contains a discussion of the extendibility of lattice orders to total orders. Finally, in §3 it is shown how Wilson's construction of lattice orders on the real field in [5] can be used to construct lattice orders on extension fields of $l$-fields.

All the proofs in this paper are based on the following representation of $l$-fields by continuous functions: By Hölder's theorem the archimedean totally ordered $o$-subfield $L$ of the $l$-field $K$ is isomorphic to a unique subfield of the reals. Identify $L$ with this subfield. Since $K$ is algebraic over $L$, the set $E_K$ of embeddings of $K$ over $L$ into $\boldsymbol{C}$ can be topologized via infinite Galois theory to become a Boolean space. Let $C(E_K)$ be the Banach algebra of continuous functions of $E_K$ into $\boldsymbol{C}$ with the norm given by $N(f) = \max(|f(\alpha)|; \alpha \in E_K)$. Define $\phi_K(x) = (\alpha(x))_{\alpha \in E_K}$ for all $x \in K$. Then $\phi_K$ embeds $K$ into $C(E_K)$ by infinite Galois theory.

After defining $e_\alpha: C(E_K) \to \boldsymbol{C}$ to be the evaluation map at $\alpha \in E_K$ and $\bar{S}$ to be the closure of the subset $S$ of a topological space, the main results of this paper can be summarized in the following

MAIN THEOREM. *For the archimedean $l$-field $K$ which is algebraic over its $o$-subfield $L$, these are equivalent*:

(1) *There is an $\alpha \in E_K$ with $\alpha(K) \subseteq \boldsymbol{R}$ such that $\overline{\phi_K(P_K)} \cap e_\alpha^{-1}(0) = 0$.*