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SWEEDLER'S TWO-COCYCLES AND HOCHSCHILD COHOMOLOGY

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For any algebra C over a commutative ring k Sweedler defined a cohomology set which generalizes Amitsur's second cohomology group $H^2(C/k)$. Any Sweedler C-two-cocycle σ gives rise to a change of rings functor $(\ )^\sigma$ from the category of C-bimodules to the category of C^σ-bimodules, where C^σ is the k-algebra with multiplication altered by σ, which in turn induces a map $\phi^n(\sigma, M)\colon H^n(C, M) \to H^n(C^\sigma, M^\sigma)$ on Hochschild cohomology for any C-bimodule M and any positive integer n. In this paper, several properties of $\phi^n(\sigma, M)$ are derived, including: If C is a finite dimensional algebra over a field k, $\phi^1(\sigma, M)$ is an injection for all σ and M.

1. Introduction. In §2 we establish our notation conventions and review the basic definitions of Sweedler's two-cocycles and Hochschild cohomology. We also recall the change of rings functor $(\ )^\sigma$ associated with a Sweedler C-two-cycle σ from the category of C-bimodules to the category of C^σ-bimodules for any algebra C over a commutative ring k.

The map $\phi^n(\sigma, M)$ induced by $(\ )^\sigma$ from the nth Hochschild cohomology group $H^n(C, M)$ of C with coefficients in the C-bimodule M to $H^n(C^\sigma, M^\sigma)$ is studied in §3. This map links the multiplicative cohomology of Sweedler and Amitsur to the additive cohomology of Hochschild. We provide an example to show that this map need not be surjective but show that if σ is invertible in an appropriate sense $\phi^n(\sigma, M)$ is actually an isomorphism. In particular, if σ is an invertible (i.e., Amitsur) two-cocycle contained in a commutative subalgebra A of C, then $\phi^n(\sigma, M)$ is an isomorphism. The behavior of $\phi^n(\sigma, M)$ under base extension of k and two-cocycles equivalent to σ is considered, and several other results are derived which are useful in studying $\ker \phi^n(\sigma, M)$.

In §4 we prove that if C is a finite dimensional algebra over a field k, $\phi^1(\sigma, M)$ is injective for all σ and M; that is, σ induces an injection of the group of equivalence classes of k-derivations of C with values in M into the group of equivalence classes of k-derivations of C^σ with values in M^σ. This result is compared with Flanigan's work on Gerstenhaber's deformation theory.

2. Notation and preliminaries. Let C be an algebra over a commutative ring k and let unadorned $\otimes$ and Hom represent $\otimes_k$ and Hom_k, respectively. Denote the n-fold tensor product $C \otimes \cdots \otimes C$