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SPANNING SURFACES FOR PROJECTIVE PLANES IN FOUR SPACE

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The purpose of this paper is to investigate two questions about the complements of projective planes in S^4. We work in the piecewise linear (PL) category (see Hudson [7] for basic definitions). All embeddings are assumed to be PL locally flat. (Equivalently PL locally unknotted, see Hudson [7] page 138.) We note though, that PL locally flat codimension two embeddings are smoothable, and vice versa (see the paragraph preceding Lemma 1) so that we could work with smooth embeddings instead. The first problem we consider is the bordism problem. Of course, we are immediately faced with the well-known fact that the projective plane does not bound any 3-manifold. Progress is further hampered by Whitney's result, [14], that a smooth projective plane in E^4 does not support a normal vector field. It follows that a PL locally flat projective plane in S^4 cannot lie on the boundary of a 3-manifold in S^4 nor can it lie in the interior of a 3-manifold in S^4. The solution to this dilemma lies in the concept of a 3-manifold with singular points. We show, Theorem 2, that every PL locally flat projective plane in S^4 bounds a 3-manifold with singular points in S^4. We also show, Theorem 1, that a PL locally flat projective plane in S^4 is unknotted iff it bounds a particular 3-manifold with singular points (namely, the cone over a Moebius band). The second problem we investigate here is a mapping problem; namely, does the complement of a knotted projective plane map onto the complement of the unknotted projective plane. In Theorem 3 we give a necessary and sufficient condition for this to occur. While the condition in Theorem 3 is necessary and sufficient, we feel there are better results possible and we discuss the shortcomings of Theorem 3 and conjecture a better result.

The first theorem characterizes unknotted projective planes in a fashion analogous to the result that a PL locally flat 2-sphere in S^4 is unknotted iff it bounds a PL 3-cell. First we remark that the standard or cannonical projective plane in S^4, denoted P^2, is the one sketched in Fig. 1 and described just before Lemma 1. (See Price and Roseman [10] for a more complete discussion of this choice of cannonical embedding.) Analogous to the case for spheres, we define a PL locally flat projective plane P, in S^4 to be unknotted iff there exists a PL homeomorphism $h: S^4 \to S^4$ with $h(P) = P^2$. To describe the analogue of the 3-cell we need the concept of a cone. Briefly,