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CYCLIC VECTORS FOR $L^p(G)$

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If G is a first countable locally compact group, then $L^p(G)$ has a cyclic vector with compact support, where $1 \leq p < \infty$.

In [3] Greenleaf and Moskowitz proved the existence of cyclic vectors for the left and right regular representation of $L^2(G)$, where G is a first countable, locally compact group, see also [4] and [5]. We generalize this result to $L^p(G)$ $(1 \leqq p < \infty)$ and certain other $L^1(G)$-modules.

THEOREM. *Let G be a locally compact group.*

(i) *If G is first countable, then there exists a continuous function u on G with compact support such that the left invariant hull of u is dense in $L^p(G)$ for $1 \leqq p < \infty$. The right hull of u (for the corresponding right action of G on $L^p(G)$) is also dense in $L^p(G)$.*

(ii) *Conversely, if $1 \leqq p < \infty$ and $L^p(G)$ has a cyclic vector, then G is first countable.*

For the proof of the theorem we need two lemmas:

LEMMA 1. *Assume that H is a closed subgroup of G which is isomorphic to $\boldsymbol{R}$. If the nonzero measure μ is concentrated on a compact subset of H, then $\{f*\mu : f \in \mathscr{K}(G)\}$ is dense in $L^p(G)$ for $1 < p < \infty$.*

Proof of Lemma 1. Define q by $1/q + 1/p = 1$. If the space defined above is not dense in $L^p(G)$, there exists a nonzero continuous function $g \in L^q(G)$ such that $\langle f*\mu, g\rangle = 0$ for all $f \in \mathscr{K}(G)$, the space of continuous functions with compact support (if g is not continuous, replace g by $h*g \neq 0$, $h \in \mathscr{K}(G)$). Put $g^{\vee}(x) = g(x^{-1})(x \in G)$, then $\mu*g^{\vee} = 0$ on G. Put $\mu_1 = \Delta_G(\cdot)^{-1/q}\cdot\mu$ and for $y \in G$, $x \in H$, set $g_y(x) = g(y^{-1}x)\Delta_G(x)^{+1/q}$ (Δ_G denotes the modular function on G). By Weil's formula ([7], pp. 42-45) $g_y \in L^q(H)$ holds for a.e. $y \in G$. A short calculation shows that

$$\mu_1*g_y^{\vee}(x) = \mu*g^{\vee}(xy)\Delta_G(x)^{-1/q} \quad \text{for} \quad x \in H\,.$$

Since g is continuous we conclude that $\mu_1*g_y^{\vee} = 0$ on H. μ_1 is concentrated on a compact subset of $H = \boldsymbol{R}$ and nonzero. The Fourier transform $\hat{\mu}_1$ is an analytic function. It follows that it has at most countably many zeros. By [1] the set $\{f*\mu_1 : f \in \mathscr{K}(H)\}$ is dense in