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SQUARE-FREE AND CUBE-FREE COLORINGS OF THE ORDINALS

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We prove: Theorem 1. The class of all ordinals has a square-free 3-coloring and a cube-free 2-coloring. Theorem 2. Every kth power-free n-coloring of α can be extended to a maximal kth power-free n-coloring of β, for some $\beta \times \alpha\cdot\omega$, where $k, n \in \omega$.

Every ordinal is conceived as the set of all smaller ordinals; ω is the least infinite ordinal. By an *interval of ordinals* we mean any set $\{\delta: \beta \leqq \delta < \gamma\}$ where β and γ are ordinals; $[\beta, \gamma)$ abbreviates $\{\delta: \beta \leqq \delta < \gamma\}$. If S and T are intervals then there can be at most one order isomorphism from S onto T.

Let S be an interval of ordinals and κ be a cardinal. A *κ-coloring* of S is just a function with domain S and range included in κ. Suppose S and T are intervals of ordinals and that f is a coloring of S while g is a coloring of T. Then the coloring f of S is *similar* to the coloring g of T provided S and T are order isomorphic and $f(\alpha) = g(h(\alpha))$ for all $\alpha \in S$ where h is the unique order isomorphism from S onto T; if f and g are clear from the context we say that S is similar to T. A coloring f of the ordinal α is *square-free* if no two adjacent nonempty intervals of α are similar; it is *cube-free* if no three consecutive nonempty intervals are all similar to each other. All these notions extend naturally to the class of all ordinals.

In Bean, Ehrenfeucht, and McNulty [1] it was shown that α has a square-free 3-coloring and a cube-free 2-coloring whenever $\alpha < (2^{\aleph_0})^+$ and the question of extending this result to all ordinals was left open. This question is resolved here.

THEOREM 1. *The class of all ordinals has a square-free* 3*-coloring and a cube-free* 2*-coloring.*

If I is a class of ordinals and α_β is an ordinal for each $\beta \in I$, then $\sum_{\beta \in I} \alpha_\beta$ denotes the *ordinal sum* of the α_β's with respect to I. (See Sierpinski [2] for details.) Finite ordinal sums are written like $\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$. For each $\beta \in I$, let $\mathrm{Int}(\beta) = [\mu, \mu + \alpha_\beta)$ where $\mu = \sum_{\gamma \in J} \alpha_\gamma$ and $J = I \cap \beta$. For each $\beta \in I$, $\mathrm{Int}(\beta)$ is order isomorphic with α_β. In fact, $\sum_{\beta \in I} \alpha_\beta$ can be construed as the disjoint union of the $\mathrm{Int}(\beta)$'s as $\beta \in I$ where the intervals are given the order type of I. This means that if f_β is a κ-coloring of α_β,