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ON THE LATTICE OF ALL CLOSED SUBSPACES OF A HERMITIAN SPACE

HANS A. KELLER

The purpose of the paper is to prove the following THEOREM: Let E be a vector space over a field K with char $K \neq 2$, and let ϕ be a nondegenerate hermitian form on E. Then the lattice of all orthogonally closed subspaces of (E, ϕ) is modular if and only if E is finite dimensional.

Introduction. It is well known that the lattice of all orthogonally (=topologically) closed subspaces of a Hilbert space H is modular only if H has finite dimension (see Birkhoff—Von Neumann [1]). We shall prove here that this is true generally for vector spaces E over commutative fields K with char $K \neq 2$, supplied with nondegenerate hermitian forms ϕ: The lattice of all orthogonally closed subspaces of (E, ϕ) is modular if and only if E is finite dimensional. Nonmodularity in the infinite dimensional case is due to the fact that then there are always two closed subspaces with nonclosed sum. In a Hilbert space one can exhibit such pairs of subspaces in a constructive way (see [3]); our general case is much more involved, and their existence will follow from an indirect proof.

1. Denotations. Let E be a (left-) vector space over a commutative field K, and $\phi: E \times E \to K$ a hermitian form with respect to an automorphism $\alpha \mapsto \bar{\alpha}$ of period 2 of K. We always assume that char $K \neq 2$. We usually write (x, y) instead of $\phi(x, y)$, and we write $x \perp y$ if $(x, y) = 0$, $x, y \in E$. Let F be a subspace of (E, ϕ). The orthogonal space of F is $F^{\perp} = \{x \in E: x \perp y \text{ for all } y \in F\}$, and the radical of F is $\operatorname{rad} F = F \cap F^{\perp}$. F is called semisimple if $\operatorname{rad} F = 0$. In particular, E is semisimple if $E^{\perp} = 0$, i.e., if ϕ is nondegenerate. A subspace F is called orthogonally closed if $F = F^{\perp\perp}(=(F^{\perp})^{\perp})$. All bases of vector spaces are algebraic. F is termed euclidean if it is semisimple and admits an orthogonal basis. Semisimple subspaces of countable dimension are always euclidean (see [2]). Every $x \in E$ induces a linear form ϕ_x on F, given by $\phi_x(z) = \phi(z, x)$, $z \in F$. We let F^* denote the antispace of the dual space of F, i.e., the K-space of all linear forms $f: F \to K$, where $(f + g)(z) = f(z) + g(z)$ and $(\alpha f)(z) = \bar{\alpha} \cdot f(z)$, $f, g \in F^*$, $\alpha \in K$. If $F^{\perp} = 0$ then E can be considered as a subspace of F^*, identifying $x \in E$ with ϕ_x.

If $E = \bigoplus_{i \in I} E_i$, and $E_i \perp E_j$ for $i \neq j$, we write $E = \bigoplus^{\perp}_{i \in I} E_i$.

2. The lattice $\mathscr{L}(E, \phi)$. Let (E, ϕ) be a semisimple hermitian