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ON THE HANDLEBODY DECOMPOSITION ASSOCIATED TO A LEFSCHETZ FIBRATION

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The classical Lefschetz hyperplane theorem in algebraic geometry describes the homology of a projective algebraic manifold M in terms of "simpler" data, namely the homology of a hyperplane section X of M and the vanishing cycles of a Lefschetz pencil containing X. This paper is a first step in proving a diffeomorphism version of the Lefschetz hyperplane theorem, namely a description of the diffeomorphism type of M in terms of "simpler" data.

Let $\tilde{M}$ be the manifold obtained from M by blowing up the axis of a Lefschetz pencil. There is a holomorphic mapping $f: \tilde{M} \to \mathrm{CP}^1$ which is a Lefschetz fibration, i.e., f has only nondegenerate critical points (in the complex sense). Using the Morse function $z \to |f(z)|^2$ on $\tilde{M}-f^{-1}(\infty)$, one obtains a handlebody decomposition of $\tilde{M}-f^{-1}(\infty)$ which may be described as follows: Let $X=f^{-1}(a)$ be a regular fiber of f. Choose a system of smooth arcs $\gamma_1, \cdots, \gamma_\mu$ starting at a and ending at the critical values of f such that the γ's are pairwise disjoint except for their common initial point. The γ's are ordered such that the tangent vectors $\gamma_1'(0), \cdots, \gamma_\mu'(0)$ rotate in a counter clockwise manner. To each γ_j one may associated a "vanishing cycle", i.e., an imbedding $\phi_j: S^n \to X$ $(\dim X=2n)$ defined up to isotopy, together with a bundle isomorphism $\phi_j': \tau \to \nu$ where τ is the tangent bundle to S^n and ν is the normal bundle of S^n in X corresponding to the imbedding ϕ_j. ϕ_j' together with the well known bundle isomorphism $\tau \oplus \varepsilon \simeq \varepsilon^{n+1}$ determines a trivialization of the normal bundle of $e^{2\pi ij/\mu} \times \phi_j(S^n)$ in $S^1 \times X$. This trivialization allows one to attach a n-handle to $D^2 \times X$ with the core $e^{2\pi ij/\mu} \times \phi_j(S^n)$. If this is done for each $j, j=1, \cdots, \mu$, the resulting manifold is diffeomorphic to $\tilde{M}$-(tubular neighborhood of $f^{-1}(\infty)$).

Using the bundle isomorphism ϕ_j' and the tubular neighborhood theorem one may identify a closed tubular neighborhood T of $\phi_j(S^n)$ in X with the tangent unit disk bundle to S^n. One may then define a diffeomorphism, up to isotopy, $\delta_j: X \to X$ with support in T. δ_j is a generalization of the classical Dehn-Lickorish twist. δ_j is the geometric monodromy corresponding to the jth critical value of f. It follows that the composition $\delta_\mu \circ \cdots \circ \delta_1$ is smoothly isotopic to the identity $1_X: X \to X$. A smooth isotopy is given by a smooth arc λ in $\mathrm{Diff}(X)$ joining the identity to $\delta_\mu \circ \cdots \circ \delta_1$. The choice of λ, up to homotopy, determines the way in which one closes off $\tilde{M}$-(tubular neighborhood of $f^{-1}(\infty)$) to obtain $\tilde{M}$.