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ON THE ZEROS OF CONVEX COMBINATIONS OF POLYNOMIALS

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Given monic nth degree polynomials $P_0(z)$ and $P_1(z)$, let $P_A(z) = (1 - A)P_0(z) + AP_1(z)$. If the zeros of P_0 and P_1 all lie in a circle $\mathscr{C}$ or on a line L, necessary and sufficient conditions are given for the zeros of P_A $(0 \leq A \leq 1)$ to all lie on $\mathscr{C}$ or L. This describes certain convex sets of monic nth degree polynomials having zeros in $\mathscr{C}$ or L. If the zeros of P_0 and P_1 lie in the unit disk and P_0 and P_1 have real coefficients, then the zeros of P_A $(0 \leq A \leq 1)$ lie in the disk $|z| < \cos(\pi/2n)/\sin(\pi/2n)$. A set is described which includes the locus of zeros of $P_A(0 \leq A \leq 1)$ as P_0 and P_1 vary through all monic nth degree polynomials having all their zeros in a compact set K. When K is path-connected, this locus is exactly the set described.

Given polynomials $P_0(z)$ and $P_1(z)$, let $P_A(z)$ denote the polynomial:

$$P_A(z) = (1 - A)P_0(z) + AP_1(z) .$$

P_A is defined for any complex value of A and the zeros of $P_A(z)$ are continuous functions of A. In particular, if A is varied through the reals between 0 and 1, the locus of zeros of $P_A(z)$ is a network of paths in the plane starting at the zeros of $P_0(z)$ and terminating in the zeros of $P_1(z)$. If the degree of P_0 is higher than that of P_1 then some of the paths of zeros must tend to infinity as A tends to one. It is the aim of this note to describe these loci of zeros when P_0 and P_1 are monic, have the same degree and are constrained to have their zeros on a circle, on a line or in a disk.

First, let P_0 and P_1 be real and have their zeros in $S^1 = \{z \in C: |z| = 1\}$ where C denotes the complex numbers. The following lemma gives a necessary and sufficient condition for the locus of zeros of $P_A(z)$. $(0 \leqq A \leqq 1)$ to be contained in S^1.

LEMMA 1. *Let $P_0(z)$ and $P_1(z)$ be real monic polynomials of degree n with their zeros contained in $S^1 - \{-1, 1\}$. Denote the zeros of $P_0(z)$ by $w_1, w_2, \cdots, w_n$ and of $P_1(z)$ by $z_1, z_2, \cdots, z_n$ and assume:*

$$w_i \neq z_j \quad (1 \leqq i, j \leqq n)$$

and

$$0 < \arg(w_i) \leqq \arg(w_j) < 2\pi$$
$$0 < \arg(z_i) \leqq \arg(z_j) < 2\pi \quad (1 \leqq i < j \leqq n) .$$