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WEAK CHEBYSHEV SUBSPACES AND ALTERNATION

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Let T be a locally compact subset of R and $C_0(T)$ the space of continuous function which vanish at infinity. An n dimensional subspace G of $C_0(T)$ may possess one of the three alternation properties:

(A-1) For each $f \in C_0(T)$ which has a unique best approximation $g_0 \in G, f - g_0$ has $n + 1$ alternating peak points;

(A-2) For each $f \in C_0(T)$, there exists a best approximation $g_0 \in G$ to f such that $f - g_0$ has $n + 1$ alternating peak points;

(A-3) For each $f \in C_0(T)$ and each best approximation $g_0 \in G$ to $f, f - g_0$ has $n + 1$ alternating peak points.

In this paper, for each $i \in \{1, 2, 3\}$ we give an intrinsic characterization of those subspaces G of $C_0(t)$ which have property (A-i).

1. Introduction. The classical alternation theorem states that if G is an n dimensional Chebyshev subspace of $C[a, b]$, then for each $f \in C[a, b]$ and its unique best approximation $g_0 \in G$, the error $f - g_0$ has $n + 1$ alternating peak points. It is natural to ask whether such a result remains valid if we replace $C[a, b]$ by $C(T)$, where T is an arbitrary compact subset of the real line R or, more generally, by $C_0(T)$, where T is any locally compact subset of R. [Here $C_0(T)$ denotes the Banach space of all real-valued contionuous functions f on T "vanishing at infinity" (i.e., $\{t \in T \mid |f(t)| \geqq \varepsilon\}$ is compact for each $\varepsilon > 0$), and endowed with the supremum norm: $\|f\| = \sup_{t \in T} |f(t)|$. When T is actually compact, we often write $C(T)$ for $C_0(T)$.] And if such a result is not valid, characterize those n dimensional subspaces G of $C_0(T)$ for which the result does hold.

Properties (A-1) and (A-2) above, in the special case $T = [a, b]$, have been considered by Jones and Karlovitz [6] who proved that an n dimensional subspace G of $C[a, b]$ has property (A-1) if and only if G has property (A-2) if and only if G is "weak Chebyshev" (i.e. G has property (W-4) defined below). Furthermore, Handscomb, Mayers, and Powell [5; Theorem 8] showed that an n dimensional subspace G of $C[a, b]$ has property (A-3) (if and) only if G is a Chebyshev subspace. (The "if" part is just the classical alternation theorem.)

In this paper, for each $i \in \{1, 2, 3\}$, we give *intrinsic characterizations* of theose subspaces G of $C_0(T)$ which have property (A-i).

It turns out that, contrary to the case when $T = [a, b]$, properties (A-1) and (A-2) are *not* the same in general; and property (A-3) does