PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 1, 1980

A NOTE ON GAP-FREQUENCY PARTITIONS

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George Andrews has introduced gap-frequency partitions in order to interpret the Rogers-Selberg q-series identities related to the modulus seven. In this paper, we give a direct derivation of the generating function for such partitions. Our approach makes it much easier to extend and generalize the notion of gap-frequency partitions.

L. J. Rogers is known today primarily for his discovery of the Rogers-Ramanujan identities:

$$(1) \qquad \prod_{\substack{n=1 \\ n\neq 0 \ \pm 2 \ (\mathrm{mod}\ 5)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m},$$

$$(2) \qquad \prod_{\substack{n=1 \\ n\neq 0 \ \pm 1 \ (\mathrm{mod}\ 5)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q)_m},$$

where $(a)_\infty = (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$,

$$(a)_m = \frac{(a)_\infty}{(aq^m)_\infty}.$$

These analytic identities came to prominence largely because of P. A. MacMahon's combinatorial interpretation of them:

(3) For $r = 1$ or 2, and any positive integer n, the partitions of n into parts not congruent to 0, $\pm r$ mod 5 are equinumerous with the partitions of n into parts with difference at least two between parts, and in which one appears as a part at most $r-1$ times.

Statement (3) can be proved from equations (1) and (2) by viewing each side of the equations as a generating function (see [3], § 19.13).

It is less well known that Rogers also discovered similar identities for the modulus 7:

$$(4) \qquad \prod_{\substack{n=1 \\ n\neq 0,\ \pm 1 \ (\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{2m^2+2m}}{(q^2;q^2)_m}(-q^{2m+2})_\infty$$

$$(5) \qquad \prod_{\substack{n=1 \\ n\neq 0,\ \pm 2 \ (\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{2m^2+2m}}{(q^2;q^2)_m}(-q^{2m+1})_\infty$$

$$(6) \qquad \prod_{\substack{n=1 \\ n\neq 0,\ \pm 3 \ (\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1} = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q^2;q^2)_m}(-q^{2m+1})_\infty .$$