PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 2, 1980

INTEGRAL FORMULAS AND INTEGRAL TESTS FOR SERIES OF POSITIVE MATRICES

AUDREY TERRAS

The main results of this paper are integral formulas which generalize that used by Siegel to prove the Minkowski-Hlawka theorem in the geometry of numbers. The main application is the derivation of an integral test for Dirichlet series of several complex variables defined by sums over integer matrices. Such an integral test yields an easy proof of the convergence of Eisenstein series, whose analytic continuations are important in harmonic analysis on Minkowski's fundamental domain for the positive $n \times n$ real matrices modulo $n \times n$ integer matrices of determinant ± 1 (i.e., $O(n)\backslash GL(n, \boldsymbol{R})/GL(n, \boldsymbol{Z})$). These integral tests can also be used to analyze the analytic continuation of Eisenstein series as sums of higher dimensional incomplete gamma functions.

For example, the easiest case of the integral formulas (that due to Siegel) implies Theorem 1, which says that for every s in the interval $(0, n/2)$ there is a positive $n \times n$ matrix Y such that the Epstein zeta function of Y and s takes on any sign. Epstein's zeta function is the simplest of the Eisenstein series for $GL(n)$. Theorem 8 gives modified incomplete gamma expansions of Eisenstein series using a method of Selberg involving invariant differential operators on the space of $n \times n$ positive matrices. Our formulas (given in Theorems 4 and 5) can be applied to show that the expansion in Theorem 8 only provides an analytic continuation of the Eisenstein series in the last complex variable. The integral formulas given here can also be used to prove Theorems 3 and 6, which generalize the Minkowski-Hlawka result on the size of the minima of quadratic forms over the integer lattice.

The *general linear group* $GL(n, D)$ for a domain D consist of all invertible $n \times n$ matrices A such that A and A^{-1} have entries in D. The symmetric space for $GL(n, \boldsymbol{R})$ is the homogeneous space $O(n)\backslash GL(n, \boldsymbol{R})$, where $O(n)$ is the orthogonal group. This homogeneous space can be identified with $\mathscr{P}_n$, *the space of all positive definite symmetric $n \times n$ real matrices*, via the map sending the coset $O(n)g$ to the matrix $Y = {}^t gg$ for g in $GL(n, \boldsymbol{R})$. Here ${}^t g$ denotes the transpose of g. References for the general theory of such symmetric spaces are [16] and [44]. Note that A in $GL(n, \boldsymbol{R})$ acts on Y in $\mathscr{P}_n$ via $Y[A] = {}^t AYA$. There have been many applications of analysis on $\mathscr{P}_n$ in physics (cf. [6]), statistics (cf. [10] and [18]), and number theory (cf. [8], [14], [26], [33], [35], [44]).