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EFFECTIVE DIVISOR CLASSES AND BLOWINGS-UP OF P^2

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Let $X_n \xrightarrow{\pi} P^2$ be the monoidal transformation of the (complex) projective plane centered at distinct points $P_1, \cdots, P_n$ of P^2. We recall that the Néron-Severi group of X_n is freely generated by the divisor class $[L]$ of the proper transform L of a line in P^2 and by the classes $[E_i]$ of the "exceptional" fibers E_i over P_i; the intersection pairing is given by

$$[L]^2=1\,; \quad [L]\cdot[E_i]=0\,; \quad [E_i]\cdot[E_j]=-\delta_{i,j}\,.$$

Let $\mathscr{M}(X_n)$ denote the monoid of elements F in the Néron-Severi group with the property that F contains an effective divisor. In this paper we

(1) construct a finite generating set for $\mathscr{M}(X_n)$ for $n \leqq 8$, and give a particularly simple geometric description of the generators when $P_1 \cdots P_n$ are in "general position";

(2) show that, for $n \geqq 9$, $\mathscr{M}(X_n)$ need not be finitely generated, despite the finite generation of the whole Néron-Severi group;

(3) prove the related result that if a nonsingular surface X contains an infinite number of exceptional curves of the first kind, then X is necessarily rational.

We will let K_{X_n} denote the cannonical class on X_n; it is given by $K_{X_n} = \pi^* K_{P^2} + \Sigma[E_i] = -3[L] + \Sigma[E_i]$. We observe that, for $n \leqq 9$, the anti-cannonical class $-K_{X_n}$ contains an effective divisor (which will also be denoted by $-K_{X_n}$ when no confusion is possible), since $H^0(X_n, \check{\omega}_{X_n})$ can be regarded as the (complex) vector space of homogeneous forms in 3 variables of degree 3 vanishing at the points $P_1 \cdots P_n$.

LEMMA 1. *Let X be any nonsingular rational surface, and let C be a curve on X with $p_a(C) \geqq 1$. Then $[C] + K_X$ is an effective class.*

Proof. The short exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{O}_X(-C) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_C \longrightarrow 0$$

yields, using Serre-duality and the rationality of X, $\dim H^0(X, \mathscr{O}_X(C) \otimes \omega_X) = \dim H^2(X, \mathscr{O}_X(-C)) = \dim H^1(C, \mathscr{O}_C) = p_a(C)$.

Recall that, for $n \leqq 8$, the points $P_1 \cdots P_n$ of P^2 are in *general*