PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 2, 1980

MODULAR SUBLATTICES OF THE LATTICE OF VARIETIES OF INVERSE SEMIGROUPS

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Kleiman used the variety $\mathscr{GP}$ of all groups to define two endomorphisms $\varphi^{\mathscr{GP}}$ and $\varphi_{\mathscr{GP}}$ of the lattice $\mathscr{L}(\mathscr{I})$ of varieties of inverse semigroups as follows: $\varphi^{\mathscr{GP}}(\mathscr{V}) = \mathscr{GP} \vee \mathscr{V}$ and $\varphi_{\mathscr{GP}}(\mathscr{V}) = \mathscr{GP} \wedge \mathscr{V}$. This introduced two congruences ν_1 and ν_2 on $\mathscr{L}(\mathscr{I})$ which have been very important in recent studies of $\mathscr{L}(\mathscr{I})$.

This paper is devoted to studying further properties of the ν_1 and $\nu_3 = \nu_1 \cap \nu_2$ congruence classes.

The first main result establishes that each ν_1-class is a complete modular sublattice of $\mathscr{L}(\mathscr{I})$, although, in some cases, the class may just consist of a single element.

It is not difficult to see that each ν_3-class has a minimum member. On the other hand, it is shown that not all ν_3-classes have maximum members. However, it is established that a large class of ν_3-classes do have maximum members. If $\mathscr{U}$ is a variety satisfying an identity of the form $x^{n+1}tt^{-1}x^{-n-1} = x^n tt^{-1}x^{-n}$ then the ν_3-class containing $\mathscr{U}$ has a maximum member. The condition that a variety satisfies this identity is equivalent to a member of conditions, one being that every member of $\mathscr{V}$ is completely semisimple and such that $\mathscr{H}$ is a congruence.

The nature of the maximum element in these cases is very interesting. If $\mathscr{U}$ satisfies the above identity, then the fundamental inverse semigroups contained in $\mathscr{U}$ constitute a variety, $\mathscr{V}$ say. Letting $\mathscr{G} = \mathscr{GP} \cap \mathscr{U}$, the maximum element in the ν_3-class containing $\mathscr{U}$ is shown to be the Mal'cev product $\mathscr{G} \circ \mathscr{V}$ of the varieties $\mathscr{G}$ and $\mathscr{V}$. It is shown that this is not valid in general. Other properties of the Mal'cev product are obtained.

1. Notation and terminology.

We shall adopt the basic notation and terminology for semigroups from [2] while, for basic results in the theory of varieties of groups, the reader is referred to [10].

The variety of all inverse semigroups, (groups, abelian groups) will be denoted by $\mathscr{I}(\mathscr{GP}, \mathscr{AGP})$ and the trivial variety by $\mathscr{T}$. Throughout the paper the term variety, if unqualified, will always mean a variety of inverse semigroups.

We will denote by $F_X(G_X)$ the free inverse semigroup (group) on a countable set X.

For any semigroup S, $\mathscr{V}(S)$ will denote the variety generated by S. For any variety $\mathscr{V}$, $F(\mathscr{V})$ will denote the relatively free inverse semigroup in $\mathscr{V}$ of countable rank and $\rho(\mathscr{V})$ will denote the