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INVARIANT SUBSPACE LATTICES AND COMPACT OPERATORS

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A general question is what can the invariant subspaces of a compact operator look like. To obtain information about this, we examine certain lattices of subspaces of a separable Hilbert space with the intent of determinig whether such a lattice could be left invariant by a compact operator.

Identify a lattice of subspaces with the lattice of projections onto those subspaces and consider only those lattices which are commutative and closed in the strong operator topology. The subclass of lattices examined are those which are multiplicity free (in the sense that the algebra generated by the projections is a maximal abelian self-adjoint algebra) and are generated as a lattice by a finite number of (mutually commuting) totally ordered sublattices. It is found that, although not all such lattices are left invariant by a compact operator, if the generating sublattices satisfy a natural independence condition, then there will be a compact, in fact Hilbert-Schmidt, operator that will leave the lattice invariant.

1. Introduction. Let $\mathscr{H}$ be a separable Hilbert space. If P and Q are two projections on $\mathscr{H}$ then their meet, $P \wedge Q$, is the projection onto the intersection of their ranges and their join, $P \vee Q$, is the projection onto the subspace union of their ranges. These operations along with the partial ordering of inclusion of ranges enables us to talk of lattices of projections. The term *subspace lattice* will refer to a lattice of projections on $\mathscr{H}$ which is closed in the strong operator topology and contains 0, the zero projection, and I, the identity projection. We restrict our attention to commutative subspace, lattices. We will call a totally ordered subspace lattice a *chain*.

Ringrose has shown that, given any chain on $\mathscr{H}$, there is a compact (in fact rank 1) operator leaving the chain invariant ([5], Lemma 3.3). The question arises of what other possible (commutative) subspace lattices are left invariant by some compact operator. A natural class of subspace lattices to consider are those generated by a finite number of mutually commuting chains.

We first note that not all such lattices can be left invariant by a compact operator. Let $\mathscr{C}$ be a chain of projections on $\mathscr{H}$ and consider the subspace lattice $\mathscr{B}$ generated by the two chains $\mathscr{C}$