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COMPACT ENDOMORPHISMS OF BANACH ALGEBRAS

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Let T be a compact endomorphism of a commutative semi-simple Banach algebra B. This paper discusses the behavior of the adjoint T^* of T on the set X' of multiplicative linear functionals on B. In particular it is shown that $\cap T^{*n}(X')$ is finite, thus generalizing the example of compact endomorphisms of the disc algebra.

0. Introduction and preliminaries. In this paper we discuss maps which are simultaneously endomorphisms of Banach algebras and compact operators. That is, these operators T are linear, satisfy $T(fg) = (Tf)(Tg)$ for all f and g in the algebra and map bounded sets into sequentially compact sets.

As a motivating example, consider the disc algebra A, the sup-norm algebra of functions analytic on the open unit disc D and continuous on $\bar{D}$. Every nonzero endomorphism T of A has the form $Tf = f \circ \varphi$ for $f \in A$, where $\varphi \in A$ and φ maps $\bar{D}$ into $\bar{D}$. It was shown in [3] that if φ is not a constant function, then T is compact if, and only if, $|\varphi(z)| < 1$ for all $z \in \bar{D}$. Moreover, for such φ, if φ_n denotes its nth iterate, then $\cap \varphi_n(\bar{D}) = \{z_0\}$ for some $z_0 \in D$, and further the spectrum $\sigma(T)$ of T satisfies $\sigma(T) = \{(\varphi'(z_0))^n | n$ is a positive integer$\} \cup \{0, 1\}$. When φ is a constant function, the range of T is one-dimensional and T is compact with $\sigma(T) = \{0, 1\}$.

We will now consider compact endomorphisms of other Banach algebras and study to what extent the properties of compact endomorphisms of the disc algebra can be generalized. Our principal results will describe the behavior of the adjoint T^* of T on the maximal ideal space of the algebra.

We first introduce some notation and terminology. Let B be a commutative semi-simple Banach algebra with unit 1 and maximal ideal space X and, in addition, let θ denote the zero functional on B. If $0 \neq T$ is a (necessarily) bounded endomorphism of B, then the adjoint T^* induces a continuous function φ from $X' \equiv X \cup \{\theta\}$ into itself in the following way. For $x \in X$, let $e_x \in B^*$ satisfy $e_x(f) = \hat{f}(x)$, where $f \to \hat{f}$ is the Gelfand transformation of B. It is easy to verify that T^*e_x is multiplicative. There are two possibilities. If $T^*e_x \neq \theta$, then $T^*e_x = e_y$ for some $y \in X$ and we let $\varphi(x) = y$. For the second case, if $T^*e_x = \theta$, we let $\varphi(x) = \theta$. We also define $\varphi(\theta) = \theta$. Since φ is essentially equal to T^* restricted to the set of multiplicative linear functionals on B, φ is a continuous function from X' to X'; φ will be called the map on X or X'