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A PROBABILISTIC PROOF OF THE GARNETT-JONES THEOREM ON BMO

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I give a probabilistic proof (via Brownian motion) of the real variable Garnett-Jones Theorem, which states that there exists some constant C_n, depending only on the dimension n, such that for all $f \in \mathrm{BMO}(\boldsymbol{R}^n)$ in the John-Nirenberg class 1 we have distance $(f, L^\infty) \leq C_n$ (the distance being measured in the BMO norm).

0. Introduction.

0.1. *Statement of the theorems.* Let $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ $(n \geqq 1)$. We then say that $f \in \mathrm{BMO}(\boldsymbol{R}^n)$ if:

$$\sup_I \frac{1}{|I|} \int_I |f - f_I| dx = \|f\|_{\mathrm{BMO}} < +\infty \tag{0.1.1}$$

I in the above expression runs through all cubes with sides parallel to the axes, $|I|$ denotes the Euclidean measure of I and

$$f_I = \frac{1}{|I|} \int_I f dx .$$

It is well known that if $f \in \mathrm{BMO}$ then there exists some $\alpha > 0$ such that:

$$\sup_I \frac{1}{|I|} \int_I e^{\alpha |f - f_I|} dx < +\infty \tag{0.1.2}$$

where I runs through the same collection as above. Let us denote by $\alpha_0 = \alpha_0(f) > 0$ the supremum of all α's for which (0.1.2) holds. α_0 can then be used to estimate the distance of f from L^∞ in BMO. More precisely we have the following theorem which is due to John Garnett and Peter Jones [4].

THEOREM (G.J.). *There exist two constants $C_1, C_2 > 0$ that only depend on the dimension n such that for all $f \in \mathrm{BMO}(\boldsymbol{R}^n)$ we have*

$$\frac{C_2}{\alpha_0} \leqq \inf_{\psi \in L^\infty} \|f - \psi\|_{\mathrm{BMO}} \leqq \frac{C_1}{\alpha_0} .$$

In this note I propose to give a probabilistic proof of the above theorem that goes via Brownian motion. To state the relevant theorem from probability theory I shall need to introduce some notation.