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THE BOUNDARY MODULUS OF CONTINUITY OF HARMONIC FUNCTIONS

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Let G be a bounded domain in the complex plane and let $u(z)$ be continuous on $\bar{G}$. In this paper we study the boundary modules of continuity, $\tilde{\omega}(\delta)$, of u on ∂G and the modulus of continuity, $\omega(\delta)$, of u on $\bar{G}$. We investigate the extent to which the inequality "$\omega(\delta) \leq \tilde{\omega}(\delta)$" holds when u is harmonic on G and show that the precise formulation of such inequalities depends on the smoothness of ∂G.

1. Introduction. Let G be a bounded domain in the complex plane and let $u(z)$ be continuous on $\bar{G}$. The modulus of continuity (MOC) of $u(z)$ on $\bar{G}$ is the function $\omega_u(\delta, \bar{G})$ defined for $\delta \geqq 0$ by

$$\omega_u(\delta, \bar{G}) = \sup \{|u(z) - u(z')| : z, z' \in \bar{G}, |z - z'| \leqq \delta\} .$$

Thus $\omega_u(\delta, \bar{G})$ is nondecreasing and $\lim_{\delta \to 0^+} \omega(\delta) = \omega(0) = 0$. If $\bar{G}$ is, say, convex, then $\omega_u(\delta)$ is subadditive and continuous. The boundary modulus of continuity (BMOC) is denoted $\tilde{\omega}_u(\delta, \partial G)$ and defined by

$$\tilde{\omega}_u(\delta, \partial G) = \sup \{|u(\zeta) - u(\zeta')| : \zeta, \zeta' \in \partial G, |\zeta - \zeta'| \leqq \delta\} .$$

When no confusion should arise, we will simply write $\omega(\delta)$ and $\tilde{\omega}(\delta)$.

It is clear that $\tilde{\omega}_u(\delta, \partial G) \leqq \omega_u(\delta, \bar{G})(\delta \geqq 0)$, and that if $u(z)$ is simply continuous on $\bar{G}$, little more can be said. In this paper we investigate the extent to which the reverse inequality holds for $u(z)$ harmonic (or analytic) on G.

Rubel, Taylor and Shields [6, p. 31] have proved the following result for u analytic.

THEOREM. *Let G be simply connected and let $\phi(\delta)(\delta \geqq 0)$ be a continuous increasing, nonnegative subadditive function. Then for $u(z)$ analytic on G, continuous on $\bar{G}$,*

$$\tilde{\omega}(\delta) \leqq \phi(\delta) \Longrightarrow \omega(\delta) \leqq C\phi(\delta) ,$$

where C is an absolute constant, independent of G.

It can be shown that $C > 1$ is necessary.

For $u(z)$ harmonic, it is known that if $G = D = \Delta(0, 1)$ is the unit disk and $u(z)$ is harmonic on D, continuous on $\bar{D}$, then

$$(1) \qquad \omega(\delta) \leqq C\left(\log \frac{1}{\delta}\right)\tilde{\omega}(\delta) \qquad \left(0 < \delta \leqq \frac{1}{2}\right) ,$$