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WHEN IS A BIPARTITE GRAPH A RIGID FRAMEWORK?

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We find the dimension of the space of stresses for all realizations of the complete bipartite graph $K_{m,n}$ in $\boldsymbol{R}^d$. That allows us to determine the infinitesimal rigidity or infinitesimal flexibility of such frameworks. The results lead both to the generic classification of $K_{m,n}$ in $\boldsymbol{R}^d$ and to a description of the realizations which deviate from the generic behavior.

1. **Introduction.** A *framework* in $\boldsymbol{R}^d$ is a finite sequence $p_1, \cdots, p_v$ of points in $\boldsymbol{R}^d$ called *vertices* together with a nonempty set E of sets $\{i, j\}$ such that $1 \leqq i < j \leqq v$. Such a framework is a *realization* in $\boldsymbol{R}^d$ of the graph E on the abstract vertex set $\{1, \cdots, v\}$. The natural and correct tendency is to think of $p_1, \cdots, p_v$ as a set of vertices in $\boldsymbol{R}^d$; we formally define it as a sequence because we wish to allow $p_i = p_j$ even when $i \neq j$ and because we often sum over the index set $\{1, \cdots, v\}$. We refer to the segment $[p_i, p_j]$ for $\{i, j\} \in E$ as an *edge* of the framework. Note that edges may have length zero. A *stress* of a framework is a real valued function ω on E such that for each vertex p_i

$$(1) \qquad \sum_{\{j:\{i,j\}\in E\}} \omega_{\{i,j\}}(p_i - p_j) = 0 \,, \quad 1 \leqq i \leqq v \,.$$

If one thinks of a framework as a physical object whose edges are stiff rods then the scalars $\omega_{\{i,j\}}$ can be thought of as an assignment to each edge of a compression or tension, depending on whether $\omega_{\{i,j\}}$ is positive or negative. Then Equation (1) says the forces at each vertex are in equilibrium.

We study stresses of frameworks because, roughly speaking, their existence indicates that some edges are redundant. The more edges, the more likely a framework is to be infinitesimally rigid, but the larger the space of stresses, the less likely. A short but precise account of these relationships, and of the connections with generic rigidity, appears in § 4.

Frameworks realizing bipartite graphs contain no triangles. Hence the rigidity of such frameworks is noteworthy. Mathematicians and engineers knew in the 19th century that a plane realization of $K_{3,3}$ is infinitesimally rigid except when its six vertices lie on a conic. The graphs $K_{4,6}$ and $K_{5,5}$ realized in space have a chance of being generically rigid; whether they actually are was asked at the special session on rigidity at the Syracuse meeting of the American Mathematical Society in October, 1978.

In this paper, we compute the dimension of the space of stres-