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ON LINEAR FORMS AND DIOPHANTINE APPROXIMATION

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Let $\vec{x}$ be a vector in $\boldsymbol{R}^K$ and let $\Lambda_j(\vec{x})$, $j=1, 2, \cdots, J$ be J linear forms in K variables. We prove that there is a lattice point $\vec{u}$ in $\boldsymbol{Z}^K$, $\vec{u}\neq\vec{0}$, for which $|\Lambda_j(\vec{u})|$ are all small (or zero) and the components of $\vec{u}$ are not too large. The bounds that we obtain improve several previous results on this problem.

1. Introduction. Let $\Lambda_1(\vec{x}), \Lambda_2(\vec{x}), \cdots, \Lambda_J(\vec{x})$ be J linear forms in K real variables $x_1, x_2, \cdots, x_K$. We assume that $B=(b_{jk})$ is a $J\times K$ matrix with complex entries such that

$$\Lambda_j(\vec{x}) = \sum_{k=1}^{K} b_{jk}x_k$$

for $j=1, 2, \cdots, J$ and so $\vec{x}$ denotes the column vector $\begin{pmatrix} x_1 \\ \cdots \\ x_K \end{pmatrix}$. A basic problem in Diophantine approximation is to show that there exists a vector $\vec{u}=\begin{pmatrix} u_1 \\ \cdots \\ u_K \end{pmatrix}$ in the integer lattice $\boldsymbol{Z}^K$, $\vec{u}\neq\vec{0}$, such that each $|\Lambda_j(\vec{u})|$ is small while the components $|u_k|$ are not too large. Quantitative results on this problem are known with various hypotheses on the Λ_j's; the usual method of proof involves an application of the pigeonhole principle (Baker [1], Lemma 1, p. 13, Gel'fond [3], Lemma 1, p. 11, Mordell [7], Theorem 3, p. 32, Siegel [8], Stolarsky [9], Chapter 2). In the present paper we make improvements on previous results of this kind by using a generalization of Minkowski's linear forms theorem which we established in [10].

In order to state our main theorem we make the following assumptions. We suppose that the forms Λ_j are real for $j=1, 2, \cdots, p$ and that the remaining forms consist of q pairs of complex conjugate forms arranged so that $\Lambda_{p+2j-1}=\bar{\Lambda}_{p+2j}$ for $j=1, 2, \cdots, q$. Thus $J=p+2q$. We also suppose that $\alpha_k \geqq 1$ for $k=1, 2, \cdots, K$, $\beta_j>0$ for $j=1, 2, \cdots, J$, and $\beta_{p+2j-1}=\beta_{p+2j}$ for $j=1, 2, \cdots, q$.

THEOREM 1. *Let M be a positive integer and suppose that*

$$M^2\left\{\prod_{l=1}^{K} \alpha_l^{-2}\right\}\left\{\prod_{j=1}^{J}\left(1+\beta_j^{-2}\sum_{k=1}^{K}\alpha_k^2|b_{jk}|^2\right)\right\} \leqq 1 . \tag{1.1}$$

Then there exist M distinct pairs of nonzero lattice points $\pm\vec{v}_m=$