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ARITHMETIC SUMS THAT DETERMINE LINEAR CHARACTERS ON $\Gamma(N)$

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A new class of arithmetic sums is defined and used to explicitly exhibit linear characters on $\Gamma(N)$, the principal congruence subgroup of level N in $SL(2, \mathbf{Z})$. As an application of this, we get a striking result on the structure of the commutator subgroup of $\Gamma(N)$.

1. Introduction. Let $\alpha, \beta, N \in \mathbf{Z}$ with $N > 1$, $(\alpha, \beta) = 1$, and $\alpha - 1 \equiv \beta \equiv 0 (\bmod N)$. For any function $\chi: \mathbf{Z}/N\mathbf{Z} \to \mathbf{R}$ define the arithmetic sum

$$t_{N,\chi}(\alpha, \beta) = \begin{cases} \sum\limits_{\substack{k=1 \\ k \not\equiv 0 (\bmod N)}}^{\beta-1} \chi\left(\left[\frac{k\alpha}{\beta}\right] (\bmod N)\right), & \beta > 0 \\ 0 & , \ \beta = 0 \\ -t_{N,\chi}(\alpha, -\beta) & , \ \beta < 0 \end{cases}$$

where [ ] denotes the integer part.

EXAMPLE 1. With $N = 2$, $\chi(0) = 1$, and $\chi(1) = -1$ we have $t_{2,\chi}(1, 2) = \chi([1/2]) = \chi(0) = 1$; $t_{2,\chi}(5, 8) = \chi([5/8]) + \chi(15/8]) + \chi([25/8]) + \chi([35/8]) = \chi(0) + \chi(1) + \chi(3) + \chi(4) = 1 - 1 - 1 + 1 = 0$.

The principal congruence subgroup of level N in SL(2, $\mathbf{Z}$) is

$$\Gamma(N) = \{A \in \mathrm{SL}(2, \mathbf{Z}) \mid A \equiv I (\bmod N)\} .$$

After preliminary work in §2 we prove

THEOREM 1. *If $\sum_{g \in Z/NZ} \chi(g) = 0$, then the map $\Gamma(N) \to \mathbf{C}$ defined by*

$$\begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \longrightarrow \exp(it_{N,\chi}(\alpha, \beta))$$

is a linear character on $\Gamma(N)$.

See [1] for the relation of this result to modular forms, knot theory, and recent work of J. B. Wagoner on diffeomorphisms of manifolds.

In §4, as an application of Theorem 1, we get a new result on the structure of the commutator subgroup of $\Gamma(N)$.

2. Preliminaries. We develop an analytic expression for $t_{N,\chi}(\alpha, \beta)$