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THE SPECTRUM OF THE LAPLACIAN ON FORMS OVER A LIE GROUP

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Let G be a compact, semi-simple, connected and simply connected Lie group. Then the bundle of p-forms, denoted by Ω^p has a Laplacian $\Delta\colon \Omega^p \to \Omega^p$ defined by the Riemannian structure on G. Then the problem of finding the eigenforms and corresponding eigenvalues is considered in this paper. Our solution is given in terms of the representation theory of G and is contained in the following.

THEOREM 1.1. *By left translation identify $\Omega^p = L^2(G) \otimes \Lambda^p \mathfrak{g}^*$ where $\mathfrak{g}$ is the Lie algebra of G. Then the spectrum of the Laplacian on p forms is given by*

(a) *the eigenvalues are*

$$c(\lambda, \mu) = \frac{1}{2}(c(\lambda) + c(\mu))$$

for $c(\lambda) = \|\lambda + \rho\|^2 - \|\rho\|^2$, λ the highest weight of an irreducible representation, ρ is half the sum of the positive roots and μ is the highest weight of an irreducible representations in the decomposition

$$\pi_\lambda \otimes \Lambda^p \operatorname{Ad}^* = \sum n_\lambda(\mu)\pi_\mu \,.$$

(b) *the corresponding eigenforms are spanned by the matrix coefficients of π_μ. Here $\pi_\mu \subset \pi_\lambda \otimes \Lambda^p \operatorname{Ad}^*$ and by the Peter-Weyl theorem we have $\Omega^p \cong \sum H_\lambda \otimes H_\lambda^* \otimes \Lambda^p \mathfrak{g}^*$ so the matrix coefficients are identified with p-forms.*

(c) *the multiplicity of $c(\lambda, \mu)$ is*

$$m(\lambda, \mu) = n_\lambda(\mu)(\dim H_\mu)^2 \,.$$

This theorem can be interpreted in the following way. Let $X_1, \cdots, X_n$ be a basis for the left invariant vector fields and $Y_1, \cdots, Y_n$ one for right invariant vector fields. Then we can define two Casimir operators, C_L using X_i and C_R using Y_i. The Theorem 1.1 can then be stated as follows.

THEOREM 1.2. *The Laplacian on p-forms is given by $\Delta = (C_L + C_R)/2$.*

It was in this form that the resnlt was first made known to the author, see [1]. The advantage of our approach over that in [1] is that we avoid long calculations in local coordinates.