PACIFIC JOURNAL OF MATHEMATICS
Vol. 90, No. 2, 1980

ASYMPTOTIC ENUMERATION OF PARTIALLY ORDERED SETS

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We define the entropy function $S(\rho)=\mathrm{Lim}_{n\to\infty}2n^{-2}\ln N(n,\rho)$, where $N(n,\rho)$ is the number of distinct partial order relations which may be defined on a set of n elements such that a fraction ρ of the possible $n(n-1)/2$ pairs are comparable. We derive upper bounds to $S(\rho)$ to show that $S(\rho)<(1/2)\ln 2$ if $\rho\geq .699$.

I. Introduction. In an earlier paper [1, hereafter referred to as I] we have studied the asymptotic enumeration of partial order relations defined over a set of n distinct objects, subject to a constraint that a given fraction ρ of the $n(n-1)/2$ pairs are comparable. Let this number be denoted by $N(n,\rho)$. We showed that $N(n,\rho)$ increases as an exponential of n^2 for large n (except in the trivial cases when ρ is either zero or one), and defined a function $S(\rho)$ by the equation

$$S(\rho)=\operatorname*{Lim}_{n\to\infty} 2n^{-2}\ln N(n,\rho)\,. \tag{1}$$

This function $S(\rho)$ may be called the entropy function as it is related to the thermodynamic entropy of a lattice-gas with a long-range three-body interaction. For details of this equivalence, the reader is referred to I.

Using upper and lower bounds on $S(\rho)$, we showed that $S(\rho)$ is a continuous function of ρ for the allowed range of ρ, $0\leqq\rho\leqq 1$. It is, however, not an analytic function of ρ. It was proved that

$$S(\rho)=\frac{1}{2}\ln 2 \qquad \text{if} \quad \frac{1}{4}\leqq\rho\leqq\frac{3}{8} \tag{2}$$

and

$$S(\rho)<\frac{1}{2}\ln 2\,, \quad \text{if} \quad \rho\leqq .083 \quad \text{or if} \quad \rho\geqq 48/49\,. \tag{3}$$

The equality in (2) could be proved, because in this range of ρ, we derived a lower bound to $S(\rho)$ which coincides with an earlier known ρ-independent upper bound due to Kleitmen and Rothschild [2]. We conjectured that the lower bound (derived in I) gives the exact value of $S(\rho)$ for all ρ. This, however, could not be proved because the corresponding upper bounds to $S(\rho)$ were too weak. We have subsequently improved the upper bounds. Using these improved bounds we can show that