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THE SCHEME OF FINITE DIMENSIONAL REPRESENTATIONS OF AN ALGEBRA

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For a finitely generated k-algebra A and a finite dimensional k-vector space M the representations of A on M form an affine k-scheme $\mathrm{Mod}_A(M)$. Of particular interest for this scheme are the connected components, the irreducible components, and the open and closed orbits under the natural action of the general linear group $\mathrm{Aut}_k(M)$, since the orbits are the equivalence classes of representations. The connected components are known for a finite dimensional algebra A. In this paper we characterize the connected components when A is commutative or an enveloping algebra of a Lie algebra in characteristic zero. For the algebra $k[x, y]/(x, y)^2$ we describe the open orbits and the irreducible components. Finally, we examine the connection with the theory of deformations of algebra representations.

Introduction. For almost any k-algebra A it is an impossible task to classify all finite dimensional A-modules. There are some standard exceptions such as the finite dimensional semisimple algebras, $A = k[x]/(x^m)$, $A = k[x, y]/(x, y)^2$, and $A = k[x]$. Even for the polynomial algebra in two indeterminates, $k[x, y]$, Gelfand and Ponomarev have shown that the classification problem is as difficult as classifying the modules over the free algebra $k\{x_1, \cdots, x_m\}$. (See [7] for details.)

If we restrict our attention to modules of a given dimension, say n, then to classify the n-dimensional A-modules is to classify the orbits of $\mathrm{Aut}_k(k^n) = \mathrm{GL}(n)$ acting on the space of n-dimensional A-module structures. If we take A to be a finitely generated k-algebra then this space of module structures is an affine k-scheme, which we call $\mathrm{Mod}_A(M)$ where $M = k^n$. Although we cannot hope to determine the orbits in most cases, we may be able to describe coarser features of $\mathrm{Mod}_A(M)$ such as the irreducible components and the connected components. It is surprising that even for the connected components there is not a complete answer for an arbitrary finitely generated k-algebra, while the irreducible components are not at all understood. Only for one interesting algebra, namely $A = k[x, y]/(x, y)^2$, have the irreducible components been determined, and for this it is essential to know the classification of the finite dimensional A-modules. (In § 5 we recall the description of the indecomposable modules from [9], determine all open orbits, and from them describe the irreducible components as found by Flanigan