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ON THE BANACH SPACES OF FUNCTIONS WITH BOUNDED UPPER MEANS

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We consider the Banach space $\mathscr{M}^p(\boldsymbol{R})$ of functions with bounded upper means. A detailed study is made of the extremal structure of the closed unit sphere, the dual space and the representations of the bounded linear functionals on $\mathscr{M}^p(\boldsymbol{R})$.

1. Introduction. In his celebrated paper on generalized harmonic analysis [13], Wiener introduced the following integrated transformation

$$(1.1)\quad s(u) = \underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{2\pi}\left(\int_{-A}^{-1} + \int_{1}^{A}\right)\frac{f(x)e^{-iux}}{-ix}dx + \frac{1}{2\pi}\int_{-1}^{1} f(x)\frac{e^{-iux}-1}{-ix}dx\ ,$$

where f is a complex valued Borel measurable function on $\boldsymbol{R}$ which satisfies $\int_{-\infty}^{\infty} |f(x)|^2/(1+x^2)dx < \infty$. By using a deep Tauberian theorem, he showed that if either limit exists, then

$$(1.2)\quad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2dx = \lim_{h\to 0+}\frac{1}{2h}\int_{-\infty}^{\infty}|s(u+h)-s(u-h)|^2du\ .$$

The formula has important applications in studying physical phenomena such as white light, noise, and turbulence where ordinary harmonic analysis is not applicable [2], [12], [13].

Unfortunately, the class $\mathscr{W}^2(\boldsymbol{R})$ of Borel measurable functions f such that $\lim_{T\to\infty} 1/2T\int_{-T}^{T}|f(x)|^2dx$ exists is not closed under addition. It is natural to consider a larger linear space which contains the above nonlinear space of functions. In [11], Marcinkiewicz defined the class $\mathscr{M}^p(\boldsymbol{R})$, $1 \leqq p < \infty$, as the set of Borel measurable functions f with

$$\|f\| = \overline{\lim_{T\to\infty}}\left(\frac{1}{2T}\int_{-T}^{T}|f(x)|^p dx\right)^{1/p} < \infty\ .$$

By identifying functions whose difference has zero norm, he proved that $(\mathscr{M}^p(\boldsymbol{R}), \|\cdot\|)$ is actually a Banach space. The space had been studied by many authors in the theory of almost periodic functions and generalized harmonic analysis (e.g., Besicovitch [4], Bohr and Følner [6], Bertrandias [3] and Lau and Lee [10]). In [10], it was shown that the transformation defined in (1.1) can be extended to an isomorphism from $\mathscr{M}^2(\boldsymbol{R})$ onto the space $\mathscr{V}^2(\boldsymbol{R})$ of functions with